

**2.7.** Consider the functions defined in equations 81 and 82. Define new functions as follows:

$$\begin{aligned}a(t) &= u(t+1) - u(t-1) \\b(t) &= u(t) - u(t-1)\end{aligned}$$

Answer the following:

1. What are the values  $a(-2)$ ,  $a(-1.5)$ ,  $a(-1)$ ,  $a(-0.5)$ ,  $a(0)$ ,  $a(0.5)$ ,  $a(1)$ ,  $a(1.5)$ ,  $a(2)$ ?
2. What are the values  $b(-2)$ ,  $b(-1.5)$ ,  $b(-1)$ ,  $b(-0.5)$ ,  $b(0)$ ,  $b(0.5)$ ,  $b(1)$ ,  $b(1.5)$ ,  $b(2)$ ?

**Solution.** We have

1.

$$\begin{aligned}a(-2) &= u((-2)+1) - u((-2)-1) = u(-1) - u(-3) = 0 - 0 = 0 \\a(-1) &= u((-1)+1) - u((-1)-1) = u(0) - u(-2) = 1 - 0 = 1 \\a(-0.5) &= u((-0.5)+1) - u((-0.5)-1) = u(0.5) - u(-1.5) = 1 - 0 = 1 \\a(0) &= u(0+1) - u(0-1) = u(1) - u(-1) = 1 - 0 = 1 \\a(0.5) &= u(0.5+1) - u(0.5-1) = u(1.5) - u(-0.5) = 1 - 0 = 1 \\a(1) &= u(1+1) - u(1-1) = u(2) - u(0) = 1 - 1 = 0 \\a(1.5) &= u(1.5+1) - u(1.5-1) = u(2.5) - u(0.5) = 1 - 1 = 0 \\a(2) &= u(2+1) - u(2-1) = u(3) - u(1) = 1 - 1 = 0\end{aligned}$$

2.

$$\begin{aligned}b(-2) &= u((-2)) - u((-2)-1) = u(-2) - u(-3) = 0 - 0 = 0 \\b(-1) &= u((-1)) - u((-1)-1) = u(-1) - u(-2) = 0 - 0 = 0 \\b(-0.5) &= u((-0.5)) - u((-0.5)-1) = u(-0.5) - u(-1.5) = 0 - 0 = 0 \\b(0) &= u(0) - u(0-1) = u(0) - u(-1) = 1 - 0 = 1 \\b(0.5) &= u(0.5) - u(0.5-1) = u(0.5) - u(-0.5) = 1 - 0 = 1 \\b(1) &= u(1) - u(1-1) = u(1) - u(0) = 1 - 1 = 0 \\b(1.5) &= u(1.5) - u(1.5-1) = u(1.5) - u(0.5) = 1 - 1 = 0 \\b(2) &= u(2) - u(2-1) = u(2) - u(1) = 1 - 1 = 0\end{aligned}$$

**2.11.** Are any of the following statements correct? In other words, in which of the following integrals can we omit the  $u(t)$  term? Explain why.

$$\begin{aligned}\int_0^{10} \sin(s)u(s)ds &= \int_0^{10} \sin(s)ds ? \\ \int_{-10}^{10} \sin(s)u(s)ds &= \int_{-10}^{10} \sin(s)ds ? \\ \int_{-10}^{-5} \sin(s)u(s)ds &= \int_{-10}^{-5} \sin(s)ds ? \\ \int \sin(s)u(s)ds &= \int \sin(s)ds ?\end{aligned}$$

**Solution.** Since the interval  $[0, 10]$  contains only positive values, we can omit the  $u(t)$  term from the first interval (it is *always* equal to 1). We cannot omit it from the second and third integrals, since for negative values of  $t$ ,  $u(t) = 0$ . For the same reason we cannot omit it from the indefinite integral either, since the region of integration may/may not contain negative times. Therefore,

$$\begin{aligned}\int_0^{10} \sin(s)u(s)ds &= \int_0^{10} \sin(s)ds \\ \int_{-10}^{10} \sin(s)u(s)ds &\neq \int_{-10}^{10} \sin(s)ds \\ \int_{-10}^{-5} \sin(s)u(s)ds &\neq \int_{-10}^{-5} \sin(s)ds \\ \int \sin(s)u(s)ds &\neq \int \sin(s)ds\end{aligned}\tag{83}$$


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**2.13.** Calculate the following integrals

$$\begin{aligned} A_1 &= \int_0^{\infty} \delta(t)u(t)dt \\ A_2 &= \int_{-\infty}^{\infty} \delta(t)u(t)dt \\ A_3 &= \int_0^{\infty} \delta(t)e^{-4t} \sin(2\pi 60t)dt \\ A_4 &= \int_{-\infty}^{\infty} [\delta(t-5) + \delta(t+1)]e^{-10t}dt \end{aligned}$$

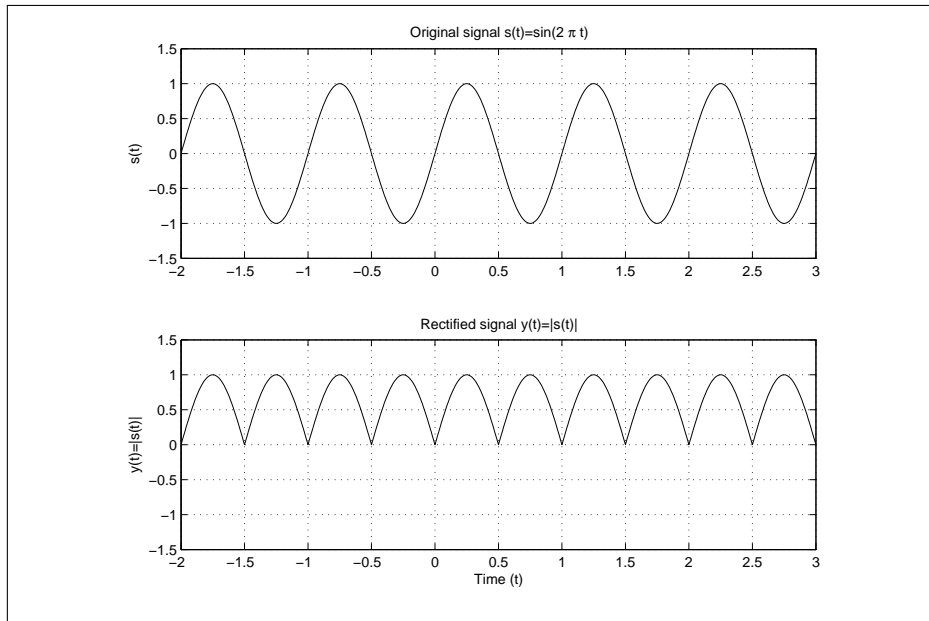
**Solution.** To calculate  $A_1$ , we need to apply equation 2.19, page 61, with  $g(t) = u(t)$ . To calculate  $A_2$ , we need to apply equation 2.18, page 61, with  $g(t) = u(t)$ . To calculate  $A_3$ , we need to apply equation 2.19, page 61, with  $g(t) = e^{-4t} \sin(2\pi 60t)$ . To calculate  $A_4$ , we need to apply equation 2.20, page 62, with  $g(t) = e^{-10t}$ , and  $t_0 = 5, -1$ . Therefore,

$$\begin{aligned} A_1 &= \int_0^{\infty} \delta(t)u(t)dt \\ &= u(0) = 1 \\ A_2 &= \int_{-\infty}^{\infty} \delta(t)u(t)dt \\ &= u(0) = 1 \\ A_3 &= \int_0^{\infty} \delta(t)e^{-4t} \sin(2\pi 60t)dt \\ &= e^{-4 \cdot 0} \sin(2\pi 60 \cdot 0) = 1 \cdot 0 = 0 \\ A_4 &= \int_{-\infty}^{\infty} [\delta(t-5) + \delta(t+1)]e^{-10t}dt \\ &= e^{-10 \cdot 5} + e^{-10 \cdot (-1)} = e^{-50} + e^{10} \approx e^{10} \end{aligned}$$


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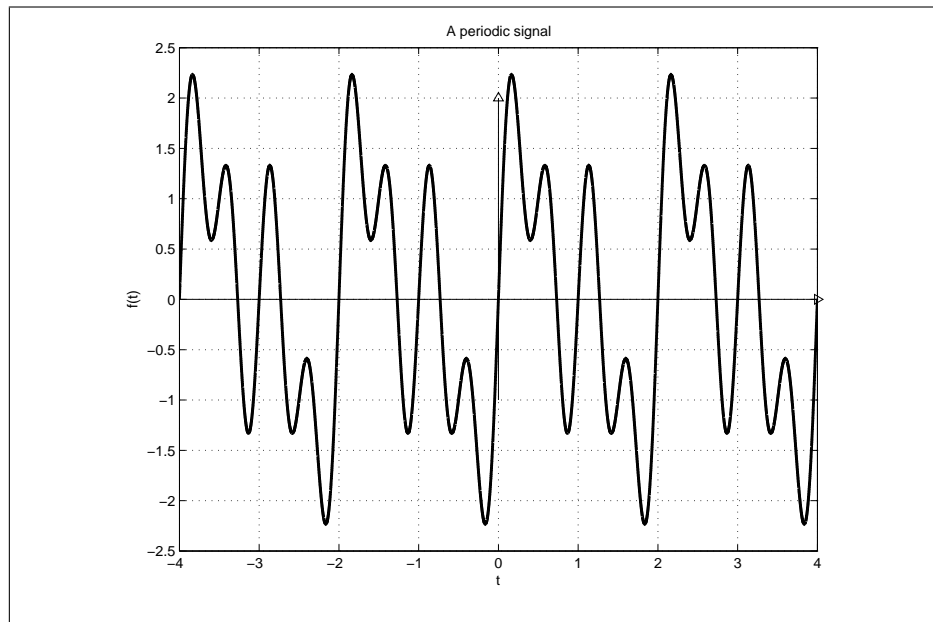
**2.18.** Let  $s(t) = \sin(2\pi t)$ . The period of this signal is  $T = 1$ . Let  $y(t) = |s(t)|$ ;  $y(t)$  is called the *rectified* sinusoidal signal. It appears frequently in power conversion systems. Find the period of  $y(t)$ .

**Solution.** The period of  $y(t)$  is  $\frac{1}{2}$ , as we can see from Figure 23.



**Figure 23:** The rectified sinusoidal signal in problem 2.18.

**2.22.** Consider the signal shown in Figure 26. Determine its period by inspection.



**Figure 26:** A periodic signal.

**Solution.** By inspection, the signal repeats itself every 2 time units. So the period is 2.

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**2.34.** Let

$$s(t) = \begin{cases} 2, & t_{low} \leq t \leq t_{high}; \\ 0, & \text{otherwise.} \end{cases}$$

The duration  $d_s$  of  $s(t)$  is  $d_s = t_{high} - t_{low}$ . What is the duration of  $s(at)$  as a function of  $a, d_s$ ?

**Solution.** The duration is  $d_s/a$ .

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**4.8.** For each one of the following complex numbers, determine whether the format is Cartesian, exponential or neither.

- $3, -3, j, -3 + 4j, -j, 2j, -3j, 3j + 2$
- $e^{j\pi/3}, e^{-\pi/4}, e^{-j\pi/4}, 5e^{-j\pi/6}, -e^{-j\pi/4}$

**Solution.**

Recall that if a complex number is to be expressed in a *Cartesian format*, it must be written in the form:

$$a + bj$$

where  $a$  must be a real number, and  $b$  must be a real number. Note that non-negative, zero or positive values are allowed for  $a$  and  $b$ . Note also that, in this format,  $j$  cannot appear with any exponents other than 1.

On the other hand, for a complex number to be expressed in a *exponential format*, it must be written in the form:

$$r \cdot e^{j\theta}$$

where  $r$  must be a nonnegative real number, and  $\theta$  must be a real number. Note that zero values are allowed for  $r$ . Note also that nonnegative, zero or positive values are allowed for  $\theta$ . Note finally that,  $j$  must appear in the exponential term.

Therefore, we can determine the following:

- (1) 3: Cartesian ( $3 = 3 + 0j$ ) or exponential ( $3 = 3 \cdot e^{j0}$ )
- (2) -3: Cartesian only (even though  $-3 = -3 \cdot e^{j0}$ ,  $r$  cannot be equal to -3.)
- (3) j: Cartesian only
- (4) -3+4j: Cartesian
- (5) -j: Cartesian
- (6) 2j: Cartesian
- (7) -3j: Cartesian
- (8) 3j+2: Cartesian
- (9)  $e^{j\pi/3}$ : exponential ( $r = 1, \theta = \pi/3$ )
- (10)  $e^{-\pi/4}$ : since  $e^{-\pi/4}$  is a real number greater than 0, it is both Cartesian ( $a = e^{-\pi/4}, b = 0$ ) and exponential ( $r = e^{-\pi/4}, \theta = 0$ )
- (11)  $e^{-j\pi/4}$ : exponential

- (12)  $5e^{-j\pi/6}$ : exponential
  - (13)  $-e^{-j\pi/4}$ : neither.
-

**\*4.12.** Let  $n$  be a negative integer.

- (a) Express  $j^n$  in Cartesian format.
- (b) Express  $j^n$  in exponential format.

**Solution.**

First observe a pattern that develops since  $j^{-1} = 1/j = -j$ ,  $j^{-2} = 1/j^2 = -1$ ,  $j^{-3} = 1/j^3 = j$ ,  $j^{-4} = 1$ . (The pattern is  $j^{-5} = -j$ ,  $j^{-6} = -1$ ,  $j^{-7} = j$ ,  $j^{-8} = 1$ , and so on.)

- (a.) For Cartesian format, we have:

$$j^n = \begin{cases} -j, & n \bmod 4 = 0 \\ -1, & n \bmod 4 = -1 \\ j, & n \bmod 4 = -2 \\ 1, & n \bmod 4 = -3 \end{cases} \quad (177)$$

- (b.) For exponential format, we have:

$$j^n = \begin{cases} e^{j\frac{3}{2}\pi}, & n \bmod 4 = 0; \\ e^{j\pi}, & n \bmod 4 = -1; \\ e^{j\frac{\pi}{2}}, & n \bmod 4 = -2; \\ e^{0j}, & n \bmod 4 = -3; \end{cases} \quad (178)$$


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**4.14.** Let  $z_1 = 2e^{j\pi/3}$  and  $z_2 = 3e^{j\pi/4}$ . Compute the following complex numbers and write your answer in exponential form.

$$\begin{aligned} z_3 &= z_1 + z_2 \\ z_4 &= z_1 - z_2 \\ z_5 &= z_1 * z_2 \\ z_6 &= z_1 / z_2 \\ z_7 &= z_2 / z_1 \\ z_8 &= z_1^2 \\ z_9 &= z_1^{-2} \\ z_{10} &= 3z_1 + 2z_2 \end{aligned}$$

Check your answers in Matlab.

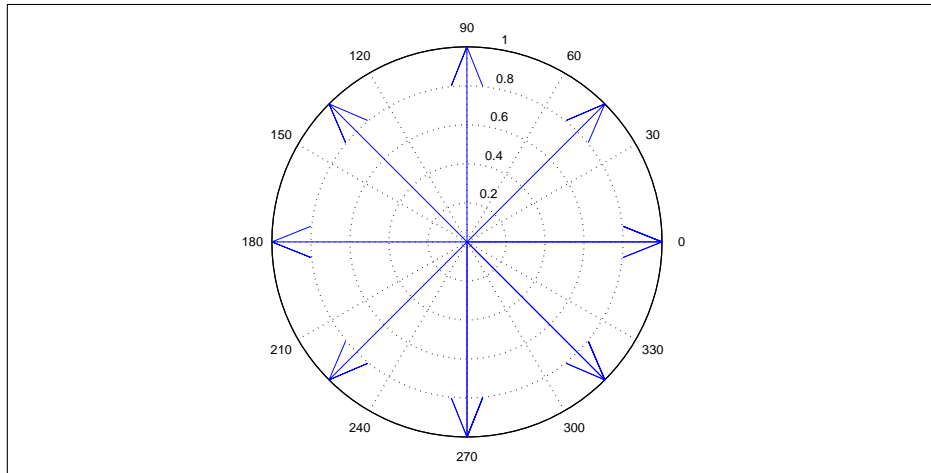
**Solution.**

Employ the Cartesian format for addition and subtraction and the exponential format for multiplication and division. Also while computing the arctan ensure that the angle is in the correct quadrant.

$$\begin{aligned} z_1 &= 2e^{j\frac{\pi}{3}} = 2 \left( \cos\left(\frac{\pi}{3}\right) + j \sin\left(\frac{\pi}{3}\right) \right) \\ &= 2 \left( \frac{1}{2} + j \frac{\sqrt{3}}{2} \right) = 1 + j \, 1.7321 \end{aligned}$$

$$\begin{aligned} z_2 &= 3e^{j\frac{\pi}{4}} = 3 \left( \cos\left(\frac{\pi}{4}\right) + j \sin\left(\frac{\pi}{4}\right) \right) \\ &= 3 \left( \frac{1}{\sqrt{2}} + j \frac{\sqrt{1}}{\sqrt{2}} \right) = 2.1213 + j \, 2.1213 \end{aligned}$$

1.  $z_3 = z_1 + z_2 = (1 + j \, 1.7321) + (2.1213 + j \, 2.1213) = 3.1213 + j \, 3.8534$   
 $= \sqrt{3.1213^2 + 3.8534^2} e^{j \arctan \frac{3.8534}{3.1213}} = 4.9589 e^{j0.89}$
2.  $z_4 = z_1 - z_2 = (1 + j \, 1.7321) - (2.1213 + j \, 2.1213) = -1.1213 - j \, 0.3893$   
 $= \sqrt{(-1.1213)^2 + (-0.3893)^2} e^{j \arctan \frac{-0.3893}{-1.1213}} = 1.1870 e^{-j2.8075}$
3.  $z_5 = z_1 * z_2 = 2e^{j\frac{\pi}{3}} \cdot 3e^{j\frac{\pi}{4}} = 6 e^{j(\frac{\pi}{3} + \frac{\pi}{4})} = 6 e^{j\frac{7\pi}{12}}$
4.  $z_6 = \frac{z_1}{z_2} = \frac{2e^{j\frac{\pi}{3}}}{3e^{j\frac{\pi}{4}}} = \frac{2}{3} e^{j(\frac{\pi}{3} - \frac{\pi}{4})} = \frac{2}{3} e^{j\frac{1\pi}{12}}$
5.  $z_7 = \frac{z_2}{z_1} = \frac{3e^{j\frac{\pi}{4}}}{2e^{j\frac{\pi}{3}}} = \frac{3}{2} e^{j(\frac{\pi}{4} - \frac{\pi}{3})} = \frac{3}{2} e^{-j\frac{1\pi}{12}}$
6.  $z_8 = z_1^2 = (2e^{j\frac{\pi}{3}})^2 = 2^2 (e^{j\frac{\pi}{3}})^2 = 4 e^{j\frac{2\pi}{3}}$



**Figure 148:** Problem 4.16.

**4.16.** Let  $z = e^{j\pi/4}$ . Compute  $z^{-n}$ , where  $n$  is a nonnegative integer. Can you see the pattern? Use the *compass* command to graph  $z^{-n}$ , for  $n = 0, 1, 2, \dots, 10$ .

**Solution.**

$z^n = e^{-jn\pi/4}$ . So  $z^0 = 1$ ,  $z^1 = e^{-j\pi/4}$ ,  $z^2 = e^{-j\pi/2}$ ,  $z^3 = e^{-3j\pi/4}$ ,  $z^4 = e^{-\pi} = -1$ ,  $z^5 = e^{-j5\pi/4}$ ,  $z^6 = e^{-j3\pi/2}$ ,  $z^7 = e^{-j7\pi/4}$ ,  $z^8 = e^{-j2\pi} = 1$ ,  $z^9 = e^{-j\pi/4} = z$ ,  $z^{10} = e^{-j\pi/2} = z^2 \dots$ , i.e.,  $z^n = z^{n-\frac{2\pi}{\pi/4}} = z^{n-8}$ .

Fig 148 shows the plot of  $z^n$ , for  $n = 0, 1, 2, \dots, 5$ .

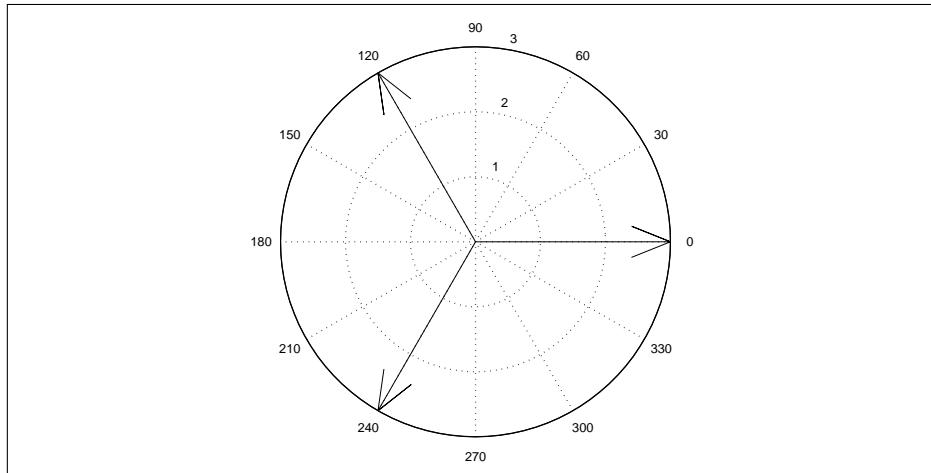
**4.18.** Find the conjugates of the following complex numbers. Express the conjugate numbers in Cartesian format.

- $j^5, j^6, j^7$ .
- $j^{-1}, j^{-2}, j^{-3}, j^{-4}, j^{-5}$ .

**Solution.**

We have

- $j^5 = j, \overline{j^5} = -j; j^6 = -1, \overline{j^6} = -1; j^7 = -j, \overline{j^7} = j$ .
  - $j^{-1} = -j, \overline{j^{-1}} = j; j^{-2} = -1, \overline{j^{-2}} = -1; j^{-3} = j, \overline{j^{-3}} = -j; j^{-4} = 1, \overline{j^{-4}} = 1; j^{-5} = -j, \overline{j^{-5}} = j$ .
-



**Figure 149:** Problem 4.29: Complex numbers  $z_k$  on the Cartesian plane.

**4.30.** Find all five solutions to the equation

$$z^5 - 32 = 0$$

Use *compass* to depict your solutions in the Cartesian plane.

**Solution.**

$z_n = 32$ . Since  $32 = |32|e^{j0}$ , in Equation 4.35 we have  $n = 5$ ,  $|a| = 32$ , and  $\theta = 0$ . Therefore, the five roots are

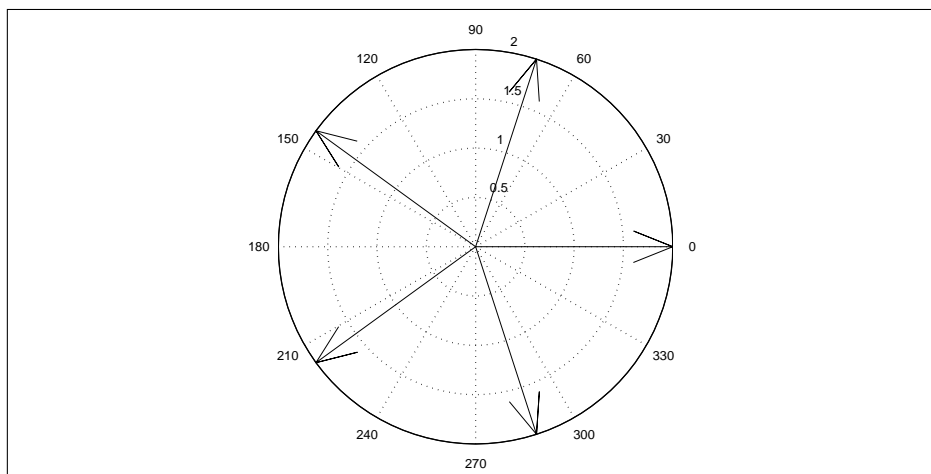
$$z_k = |32|^{\frac{1}{5}} \cdot e^{j\frac{0}{5} + jk\frac{2\pi}{5}} = 2 \cdot e^{jk\frac{2\pi}{5}}, k = 0, 1, 2, 3, 4.$$

or,

$$\begin{aligned} z_0 &= 2 \\ z_1 &= 2 \cdot e^{j2\frac{\pi}{5}} \\ z_2 &= 2 \cdot e^{j4\frac{\pi}{5}} \\ z_3 &= 2 \cdot e^{j6\frac{\pi}{5}} \\ z_4 &= 2 \cdot e^{j8\frac{\pi}{5}} \end{aligned}$$

These complex numbers can be plotted on the Cartesian plane using the following MATLAB script:

```
z1 = 2;
z2 = 2*exp(j*2*pi/5);
z3 = 2*exp(j*4*pi/5);
z4 = 2*exp(j*6*pi/5);
z5 = 2*exp(j*8*pi/5);
```



**Figure 150:** Problem 4.30 - Complex numbers  $z_k$  on the Cartesian plane.

```
compass(z1)
hold on
compass(z2)
compass(z3)
compass(z4)
compass(z5)
hold off
```

See Figure 150.

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**4.35.** The following are phasor notations. Write the corresponding signals in the time domain.

- $\tilde{v}_1 = 6\angle 0$ ; the frequency is 1000 Hz
- $\tilde{v}_2 = 3\angle(-3\pi/2)$ ; the frequency is 2 KHz

**Solution.**

1.  $\tilde{v}_1 = 6\angle 0 = 6 e^{j 0} v_1(t) = \text{Re}(6 e^{j 0} e^{j 2\pi 1000t}) = 6 \cos(2\pi 1000t)$
  2.  $\tilde{v}_2 = 3\angle(-3\pi/2) = 3 e^{j (-\frac{3\pi}{2})} v_2(t) = \text{Re}(3 e^{j (-\frac{3\pi}{2})} e^{j 2\pi 2000t}) = 3 \cos(2\pi 2000t - \frac{3\pi}{2})$
-

**4.40.** Consider the complex-valued function

$$H(\omega) = \frac{5}{1 - j2\omega}$$

Compute and write in both exponential and Cartesian form the following:

$$H(0), H(\pi), H(-\pi), H(2\pi), H(2\pi4)$$

**Solution.**

$$H(\omega) = \frac{5}{1 - j2\omega}$$

1.

$$H(0) = \frac{5}{1 - j2 \cdot 0} = 5 + j \cdot 0 = 5e^{j \cdot 0}$$

2.

$$\begin{aligned} H(\pi) &= \frac{5}{1 - j2\pi} \\ &= 5 \frac{(1 + j2\pi)}{(1 - j2\pi)(1 + j2\pi)} \\ &= 5 \left( \frac{1 + j2\pi}{1 + 4\pi^2} \right) \\ &= .123 + j0.776 = .785e^{1.414} \end{aligned}$$

3.

$$\begin{aligned} H(-\pi) &= \frac{5}{1 + j2\pi} \\ &= 5 \frac{(1 - j2\pi)}{(1 + j2\pi)(1 - j2\pi)} \\ &= 5 \left( \frac{1 - j2\pi}{1 + 4\pi^2} \right) \\ &= .123 - j0.776 = .785e^{-1.414} \end{aligned}$$

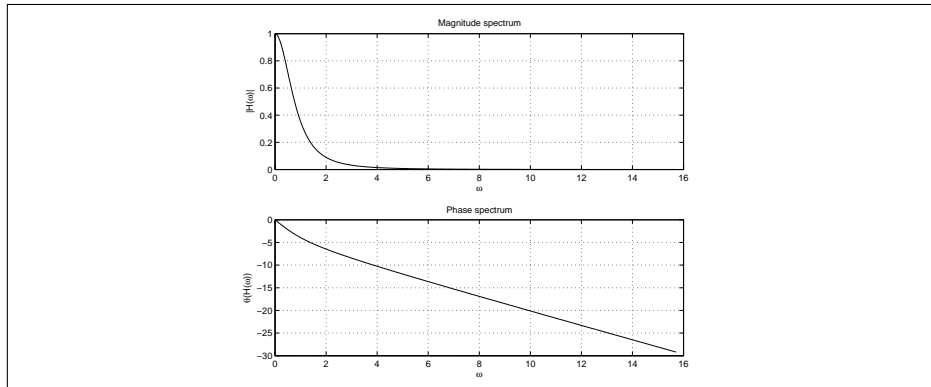
4.

$$\begin{aligned} H(2\pi) &= \frac{5}{1 - j2 \cdot 2\pi} \\ &= 5 \frac{(1 + j4\pi)}{(1 - j4\pi)(1 + j4\pi)} \\ &= 5 \left( \frac{1 + j4\pi}{1 + 16\pi^2} \right) \\ &= .0315 - j0.395 = .396e^{1.491} \end{aligned}$$

5.

$$\begin{aligned}H(2\pi 4) &= \frac{5}{1-j2.8\pi} \\&= 5 \frac{(1+j16\pi)}{(1-j16\pi)(1+j16\pi)} \\&= 5 \left( \frac{1+j16\pi}{1+256\pi^2} \right) \\&= (1.978+j99.4) * 10^{-3} = 9.88 * 10^{-3} e^{1.55}\end{aligned}$$

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**Figure 155:** Problem 4.42.

**4.42.** Consider the  $H(\omega)$  function

$$H(\omega) = \frac{e^{-j\frac{\omega}{2}}}{(1 + j\omega)^3}$$

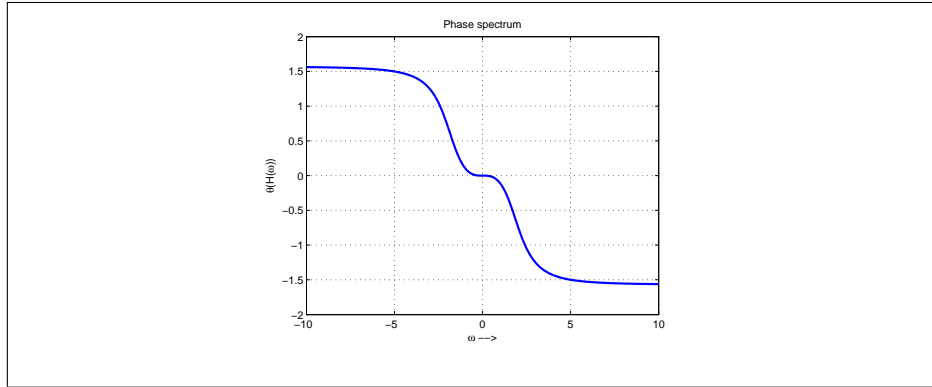
Find and plot the magnitude and phase of  $H(\omega)$ .

**Solution.**

$$|H(w)| = \frac{|e^{-j\omega/2}|}{|(1 + j\omega)^3|} = \frac{1}{(1 + \omega^2)^{3/2}}$$

$$\theta(H(w)) = \angle(e^{-j\frac{\omega}{2}}) - \angle((1 + j\omega)^3) = -\frac{\omega}{2} - 3 \arctan(\omega)$$

The matlab code titled `problem4_42.m` generates the frequency spectrum given in figure 155:



**Figure 157:** Problem 4.43, phase spectrum.

**4.44.** Consider the complex-valued function

$$S(n) = \begin{cases} 0, & n = 0 \\ \frac{1}{j\pi n} [\cos(\frac{n\pi}{5}) - \cos(\frac{3n\pi}{5})], & n \neq 0 \end{cases}$$

Find and plot the magnitude and phase spectrum.

**Solution.**

The magnitude and phase of  $S(0)$  is zero. When  $n \neq 0$ ,

$$\begin{aligned} |S(n)| &= \frac{|\cos(\frac{n\pi}{5}) - \cos(\frac{3n\pi}{5})|}{|j\pi n|} = \frac{|\cos(\frac{n\pi}{5}) - \cos(\frac{3n\pi}{5})|}{\pi|n|} \\ &= 2 \frac{|\sin(\frac{2n\pi}{5}) \sin(\frac{n\pi}{5})|}{\pi|n|} \\ \theta(S(n)) &= \angle[\cos(\frac{n\pi}{5}) - \cos(\frac{3n\pi}{5})] - \angle(j\pi n). \end{aligned}$$

Note that  $\cos(\frac{n\pi}{5}) - \cos(\frac{3n\pi}{5})$  is a real number for any value of  $n$  and  $j\pi n$  is an imaginary number for any value of  $n$ .

Therefore, the phase of  $\cos(\frac{n\pi}{5}) - \cos(\frac{3n\pi}{5})$  can be either 0, and that happens for all  $n$  such that

$$\cos(\frac{n\pi}{5}) - \cos(\frac{3n\pi}{5}) \geq 0$$

or  $\pi$ ; that happens for all  $n$  such that

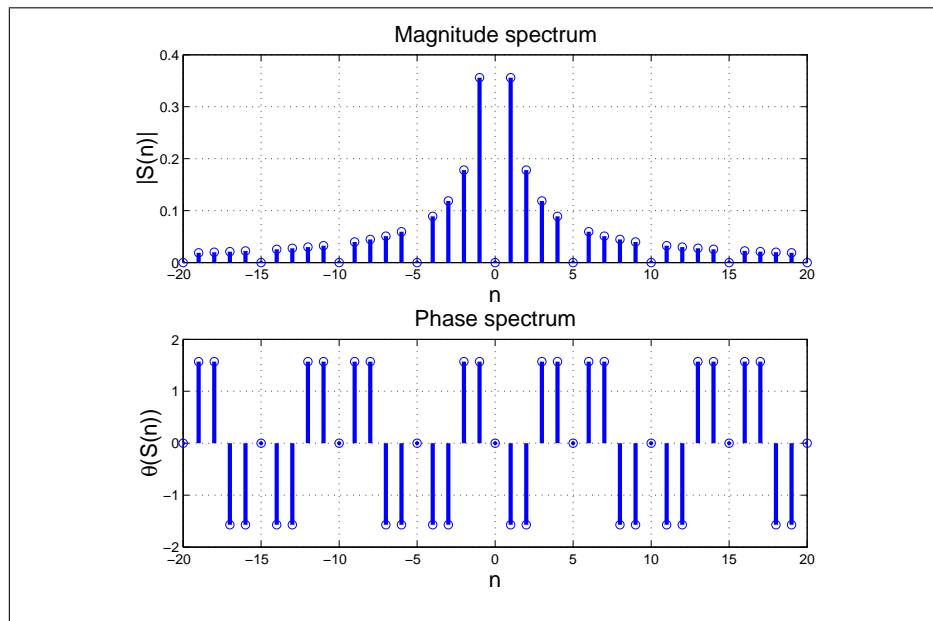
$$\cos(\frac{n\pi}{5}) - \cos(\frac{3n\pi}{5}) < 0.$$

The phase of  $j\pi n$  can be either  $\pi/2$  (that happens when  $n \geq 0$ ) or  $-\pi/2$  (that happens when  $n < 0$ ).

Here are some randomly calculated values for the phase:

$n$	$\frac{\cos(\frac{n\pi}{5}) - \cos(\frac{3n\pi}{5})}{\pi n}$	$\theta(S(n))$
0	-	0
1	0.3559	$-\frac{\pi}{2}$
2	0.1779	$-\frac{\pi}{2}$
3	-0.1186	$\frac{\pi}{2}$
4	-0.0890	$\frac{\pi}{2}$
5	0	0
6	-0.0593	$\frac{\pi}{2}$
7	-0.0508	$\frac{\pi}{2}$
8	0.0445	$-\frac{\pi}{2}$
9	0.0395	$-\frac{\pi}{2}$

In summary, the phase plot can take only three distinct values,  $\frac{\pi}{2}$ , 0 and  $-\frac{\pi}{2}$ . The corresponding frequency spectrum is illustrated in figure 158. (See Matlab code in file `problem4_44.m`.)



**Figure 158:** Problem 4.44.

**4.46.** Compute the following integral; write  $A$  in both Cartesian and exponential form.

$$A = \int_0^{\infty} e^{(-3+j4)x} dx$$

**Solution.**

We have

$$\begin{aligned} A &= \int_0^{\infty} e^{(-3+j4)x} dx \\ &= \frac{1}{-3+j4} e^{(-3+j4)x} \Big|_0^{\infty} \\ &= \frac{1}{-3+j4} (e^{(-3+j4)\cdot\infty} - e^{(-3+j4)\cdot 0}) \end{aligned}$$

For any complex number with a negative real part, say,  $-a + j b$ ,

$$\begin{aligned} e^{(-a+j b)\cdot\infty} &= \lim_{T \rightarrow \infty} e^{(-a+j b)T} = \lim_{T \rightarrow \infty} e^{-aT} \cdot e^{j bT} \\ &= \lim_{T \rightarrow \infty} e^{-aT} \cdot \lim_{T \rightarrow \infty} [\cos(bT) + j \sin(bT)] = 0, \end{aligned}$$

as the terms  $\cos(bT), \sin(bT)$  are always bounded by  $\pm 1$ . Therefore, we can write

$$\begin{aligned} A &= \frac{1}{-3+j4} (0 - 1) \\ &= \frac{1}{3-j4} \end{aligned}$$

To convert into Cartesian format, write

$$\frac{1}{3-j4} = \frac{1}{3-j4} \cdot \frac{3+j4}{3+j4} = \frac{3+j4}{3^2+4^2} = \frac{3}{25} + j \frac{4}{25}.$$

The magnitude of  $A$  is

$$|A| = \frac{1}{\sqrt{3^2 + (-4)^2}} = \frac{1}{5} = 0.2$$

and its phase is

$$\angle(A) = \angle(1) - \angle(3-j4) = 0 - \arctan(-4/3) = 0.9273$$

So, in exponential format,  $A$  is expressed as  $0.2 e^{j 0.9273}$ .

**4.49.** Let  $a, \omega$  be known, positive, real-valued constants. Compute the integral and write it in both Cartesian and exponential form.

$$A(\omega) = \int_0^{\infty} e^{-at} e^{-j\omega t} dt$$

**Solution.**

$$\begin{aligned} A(\omega) &= \int_0^{\infty} e^{-at} e^{-j\omega t} dt \\ &= \int_0^{\infty} e^{-(a+j\omega)t} dt \\ &= \left. \frac{e^{-(a+j\omega)t}}{-(a+j\omega)} \right|_0^{\infty} \\ &= \frac{-1}{a+j\omega} [0 - 1] \\ &= \frac{1}{a+j\omega} \\ &= \frac{a-j\omega}{a^2+\omega^2} \\ &= \frac{a}{a^2+\omega^2} - j \frac{\omega}{a^2+\omega^2} \\ &= \frac{1}{\sqrt{a^2+\omega^2}} e^{j \arctan(-\omega/a)} \end{aligned}$$

---

**5.3.** Find three 3-dimensional vectors that are pair-wise orthogonal.

**Solution.** Let  $\mathbf{x} = [2, 0, 0]$ ,  $\mathbf{y} = [0, 3, 0]$  and  $\mathbf{z} = [0, 0, 10]$ . Then

$$\begin{aligned}\mathbf{x} \cdot \mathbf{y} &= [2, 0, 0] \cdot [0, 3, 0] \\ &2 \cdot 0 + 0 \cdot 3 + 0 \cdot 0 = 0 \\ \mathbf{x} \cdot \mathbf{z} &= [2, 0, 0] \cdot [0, 0, 10] \\ &2 \cdot 0 + 0 \cdot 0 + 0 \cdot 10 = 0 \\ \mathbf{z} \cdot \mathbf{y} &= [0, 0, 10] \cdot [0, 3, 0] \\ &0 \cdot 0 + 0 \cdot 3 + 10 \cdot 0 = 0\end{aligned}$$

---

5.8. Let

$$\mathbf{a} = \begin{bmatrix} -1 \\ 1 \\ 0 \end{bmatrix}, \quad \mathbf{b} = \begin{bmatrix} 1 & -1 & -3 \end{bmatrix}$$

Compute the following, if possible, or state that the operation is not legal:

- $\mathbf{c}_1 = \mathbf{ab}$
- $\mathbf{c}_2 = \mathbf{ab}^T$
- $\mathbf{c}_3 = \mathbf{a}^T \mathbf{b}$
- $\mathbf{c}_4 = \mathbf{a}^T \mathbf{b}^T$

**Solution.**

$$\mathbf{a} = \begin{bmatrix} -1 \\ 1 \\ 0 \end{bmatrix}$$

$$\mathbf{b} = \begin{bmatrix} 1 & -1 & -3 \end{bmatrix}$$

$$1. \ c_1 = \mathbf{ab} = \begin{bmatrix} -1 \\ 1 \\ 0 \end{bmatrix} \begin{bmatrix} 1 & -1 & -3 \end{bmatrix} = \begin{bmatrix} -1 & 1 & 3 \\ 1 & -1 & -3 \\ 0 & 0 & 0 \end{bmatrix}$$

$$2. \ c_2 = \mathbf{ab}^T = \begin{bmatrix} -1 \\ 1 \\ 0 \end{bmatrix} \begin{bmatrix} 1 \\ -1 \\ -3 \end{bmatrix}$$

The multiplication is not legal in this case as the number of columns in  $\mathbf{a}$  are not equal to number of rows in  $\mathbf{b}^T$ .

$$3. \ c_3 = \mathbf{a}^T \mathbf{b} = \begin{bmatrix} -1 & 1 & 0 \end{bmatrix} \begin{bmatrix} 1 & -1 & -3 \end{bmatrix}$$

The multiplication is not legal in this case as the number of columns in  $\mathbf{a}^T$  are not equal to number of rows in  $\mathbf{b}$ .

$$4. \ c_4 = \mathbf{a}^T \mathbf{b}^T = \begin{bmatrix} -1 & 1 & 0 \end{bmatrix} \begin{bmatrix} 1 \\ -1 \\ -3 \end{bmatrix} = \begin{bmatrix} -2 \end{bmatrix}$$

**5.10.** Let

$$\mathbf{A} = \begin{bmatrix} 2 & 0 \\ -1 & -1 \\ 0 & -2 \end{bmatrix}, \quad \mathbf{B} = \begin{bmatrix} 1 & 1 & -1 \\ 3 & 2 & 0 \\ -1 & 1 & 0 \end{bmatrix}$$

Compute the following, if possible, or state that the operation is not legal:

- $\mathbf{C}_1 = \mathbf{AB}$
- $\mathbf{C}_2 = \mathbf{AB}^T$
- $\mathbf{C}_3 = \mathbf{A}^T \mathbf{B}$
- $\mathbf{C}_4 = \mathbf{A}^T \mathbf{B}^T$

**Solution.** We have

$$\mathbf{A} = \begin{bmatrix} 2 & 0 \\ -1 & -1 \\ 0 & -2 \end{bmatrix}$$
$$\mathbf{B} = \begin{bmatrix} 1 & 1 & -1 \\ 3 & 2 & 0 \\ -1 & 1 & 0 \end{bmatrix}$$

1.  $C_1 = AB = \begin{bmatrix} 2 & 0 \\ -1 & -1 \\ 0 & -2 \end{bmatrix} \begin{bmatrix} 1 & 1 & -1 \\ 3 & 2 & 0 \\ -1 & 1 & 0 \end{bmatrix}$

The multiplication is not legal in this case as the number of columns in A are not equal to number of rows in B.

2.  $C_2 = AB^T = \begin{bmatrix} 2 & 0 \\ -1 & -1 \\ 0 & -2 \end{bmatrix} \begin{bmatrix} 1 & 3 & -1 \\ 1 & 2 & 1 \\ -1 & 0 & 0 \end{bmatrix}$

The multiplication is not legal in this case as the number of columns in A are not equal to number of rows in  $B^T$ .

3.  $C_3 = A^T B = \begin{bmatrix} 2 & -1 & 0 \\ 0 & -1 & -2 \end{bmatrix} \begin{bmatrix} 1 & 1 & -1 \\ 3 & 2 & 0 \\ -1 & 1 & 0 \end{bmatrix} = \begin{bmatrix} -1 & 0 & -2 \\ -1 & -4 & 0 \end{bmatrix}$

4.  $C_4 = A^T B^T = \begin{bmatrix} 2 & -1 & 0 \\ 0 & -1 & -2 \end{bmatrix} \begin{bmatrix} 1 & 3 & -1 \\ 1 & 2 & 1 \\ -1 & 0 & 0 \end{bmatrix} = \begin{bmatrix} 1 & 4 & -3 \\ 1 & -2 & -1 \end{bmatrix}$

**6.27.** Consider the system of equations given below.

$$\begin{bmatrix} 2 & -3 & 4 \\ -1 & -2 & 2 \\ 1 & a & 6 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} 2 \\ b \\ 1 \end{bmatrix}$$

- (a) Solve the system in terms of  $a$  and  $b$  using Gaussian elimination.
- (b) For what values of  $a$  will the system have no unique solution?
- (c) For what combination of values for  $a$  and  $b$  will the system have no solution, that is, the system is inconsistent?
- (d) For what values of  $a$  and  $b$  will the system have multiple solutions?

**Solution.** (a) Let's denote

$$\mathbf{A} = \begin{bmatrix} 2 & -3 & 4 \\ -1 & -2 & 2 \\ 1 & a & 6 \end{bmatrix}, \mathbf{b} = \begin{bmatrix} 2 \\ b \\ 1 \end{bmatrix}$$

The augmented matrix  $[\mathbf{A}\mathbf{b}]$  would then be

$$\begin{bmatrix} 2 & -3 & 4 & 2 \\ -1 & -2 & 2 & b \\ 1 & a & 6 & 1 \end{bmatrix}$$

Performing Gaussian elimination:

Row 2  $\leftarrow$  Row 2 + Row 3

$$\begin{bmatrix} 2 & -3 & 4 & 2 \\ 0 & a-2 & 8 & b+1 \\ 1 & a & 6 & 1 \end{bmatrix}$$

Row 3  $\leftarrow$  Row 3 - 0.5  $\times$  Row 1

$$\begin{bmatrix} 2 & -3 & 4 & 2 \\ 0 & a-2 & 8 & b+1 \\ 0 & a+1.5 & 4 & 0 \end{bmatrix}$$

Row 1  $\leftarrow$  0.5  $\times$  Row 1

$$\begin{bmatrix} 1 & -1.5 & 2 & 1 \\ 0 & a-2 & 8 & b+1 \\ 0 & a+1.5 & 4 & 0 \end{bmatrix} \tag{257}$$

Row 2  $\leftarrow$  1/( $a-2$ )  $\times$  Row 2. We assume that  $a \neq 2$ .

$$\begin{bmatrix} 1 & -1.5 & 2 & 1 \\ 0 & 1 & 8/(a-2) & (b+1)/(a-2) \\ 0 & a+1.5 & 4 & 0 \end{bmatrix} \quad (258)$$

Row 3  $\leftarrow$  Row 3  $-(a+1.5) \times$  Row 2. We assume that  $a \neq -1.5$ .

$$\begin{bmatrix} 1 & -1.5 & 2 & 1 \\ 0 & 1 & 8/(a-2) & (b+1)/(a-2) \\ 0 & 0 & 4-8(a+1.5)/(a-2) & -(a+1.5)(b+1)/(a-2) \end{bmatrix}$$

Simplifying the (3,3) element, we have

$$\begin{bmatrix} 1 & -1.5 & 2 & 1 \\ 0 & 1 & 8/(a-2) & (b+1)/(a-2) \\ 0 & 0 & -(4a+20)/(a-2) & -(a+1.5)(b+1)/(a-2) \end{bmatrix} \quad (259)$$

Row 3  $\leftarrow -(a-2)/(4a+20) \times$  Row 3. We assume that  $a \neq -5$ .

$$\begin{bmatrix} 1 & -1.5 & 2 & 1 \\ 0 & 1 & 8/(a-2) & (b+1)/(a-2) \\ 0 & 0 & 1 & (a+1.5)(b+1)/(4a+20) \end{bmatrix}$$

Row 2  $\leftarrow$  Row 2  $-8/(a-2) \times$  Row 3.

$$\begin{bmatrix} 1 & -1.5 & 2 & 1 \\ 0 & 1 & 0 & (b+1)/(a-2) - 8(a+1.5)(b+1)/[(a-2)(4a+20)] \\ 0 & 0 & 1 & (a+1.5)(b+1)/(4a+20) \end{bmatrix}$$

Simplifying the (2,4) element, we have

$$\begin{bmatrix} 1 & -1.5 & 2 & 1 \\ 0 & 1 & 0 & (-ab - a + 2b + 2)/[(a-2)(a+5)] \\ 0 & 0 & 1 & (a+1.5)(b+1)/(4a+20) \end{bmatrix}$$

Row 1  $\leftarrow$  Row 1  $-2 \times$  Row 3. After simplifying the (2,4) element, we have

$$\begin{bmatrix} 1 & -1.5 & 0 & (-ab - 3a - 1.5b + 18.5)/(4a+20) \\ 0 & 1 & 0 & (-ab - a + 2b + 2)/[(a-2)(a+5)] \\ 0 & 0 & 1 & (a+1.5)(b+1)/(4a+20) \end{bmatrix}$$

Row 1  $\leftarrow$  Row 1  $+1.5 \times$  Row 2.

$$\begin{bmatrix} 1 & 0 & 0 & 1.5(-ab - a + 2b + 2)/[(a-2)(a+5)] + (-ab - 3a - 1.5b + 18.5)/(4a+20) \\ 0 & 1 & 0 & (-ab - a + 2b + 2)/[(a-2)(a+5)] \\ 0 & 0 & 1 & (a+1.5)(b+1)/(4a+20) \end{bmatrix}$$

The solution is

$$\begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} 1.5(-ab - a + 2b + 2)/[(a-2)(a+5)] + (-ab - 3a - 1.5b + 18.5)/(4a+20) \\ (-ab - a + 2b + 2)/[(a-2)(a+5)] \\ (a+1.5)(b+1)/(4a+20) \end{bmatrix}$$

and the assumptions are:  $a \neq -2$ ,  $a \neq -1.5$  and  $a \neq -5$ .  
When  $a = -2$ , the second row in equation 257 reads

$$x_3 = \frac{b+1}{8}$$

From the third row, we can find

$$x_2 = -\frac{b+1}{7}$$

and from the first one

$$x_1 = 1 - 1.5\frac{b+1}{7} + \frac{b+1}{7}$$

So, we can remove the assumption  $a \neq -2$ .

When  $a = -1.5$ , the third row in equation 258 reads

$$x_3 = 0$$

The second row yields

$$x_2 = -\frac{b+1}{3.5}$$

and from the first one

$$x_1 = 1 + 1.5\frac{b+1}{3.5}$$

So, we can also remove the assumption  $a \neq -1.5$ .

(b) When  $a = -5$ , the third row in equation 259 reads

$$0 \cdot x_3 = -\frac{3.5(b+1)}{2}$$

If  $b = -1$ , the system has multiple solutions.

(c) If  $b \neq -1$ , there is no solution, that is, the system is inconsistent.

(d) See part (b), this is the same question.

We can check this lengthy computation by running the Matlab script `problem6_27.m`.

```
% Problem 6.27
clear
syms a ;
syms b ;
% define the matrix and vector of constants
A = [2 -3 4 ; -1 -2 2 ; 1 a 6]
c = [ 2;b;1]
x = A\c
```

This script produced the output

A =

[ 2, -3, 4]  
[ -1, -2, 2]  
[ 1, a, 6]

c =

2  
b  
1

x =

$$\frac{(2a - 9b - 2ab + 11)/(4(a + 5)) - (b + 1)/(a + 5)}{((2a + 3)(b + 1))/(8(a + 5))}$$

---

**7.3.** Integrate both sides of the following equation to find  $y(t)$  for  $t \geq 0$ , with the initial condition  $y(0) = 10$ .

$$\frac{dy(t)}{dt} = f(t),$$

where

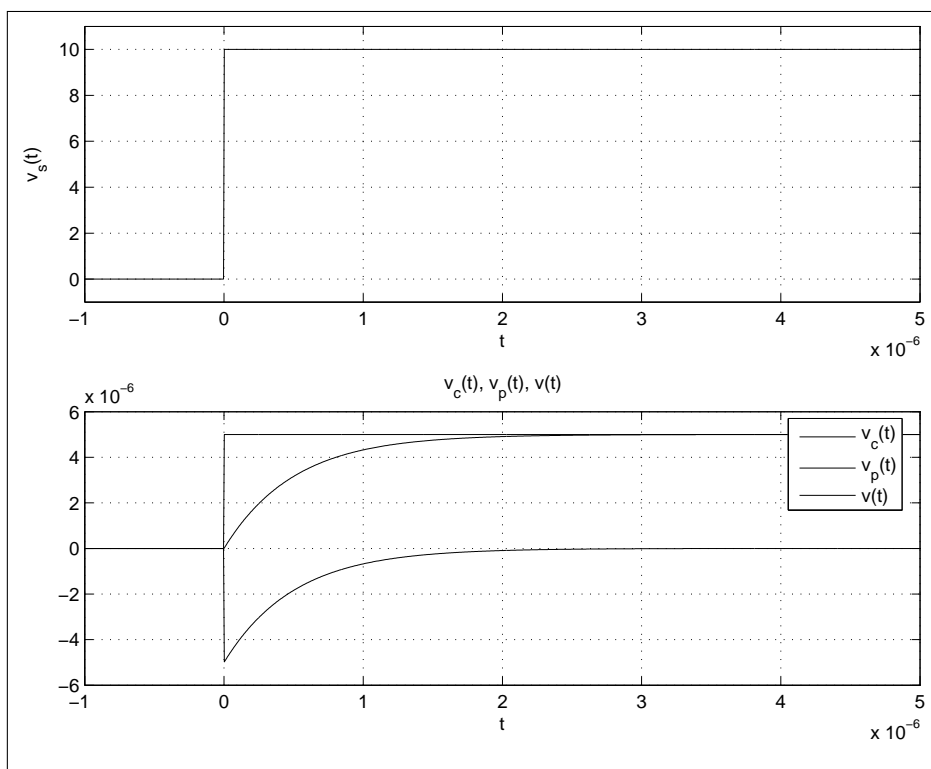
$$f(t) = \begin{cases} 0 & t < 1 \\ 2, & 1 \leq t < 2 \\ 0, & 2 \leq t < 4 \\ -1, & 4 \leq t < 6 \\ 0, & 6 \leq t \end{cases}$$

**Solution.** In this problem, we assume that  $f(t)$  is a continuous function of  $t$ .

1. For  $t < 1$ , we have  $\frac{dy(t)}{dt} = 0$ . Integrating both sides, we get  $y(t) = C_1$ .  
From the initial condition,  $y(0) = 10$ , we have  $C_1 = 10$  and thus we have  $y(1) = 10$ .
2. For  $1 \leq t < 2$ , we have  $\frac{dy(t)}{dt} = 2$ . Integrating both sides, we get  $y(t) = 2t + C_2$ .  
From the initial condition,  $y(1) = 10$ , we have  $C_2 = 8$  and thus we have  $y(2) = 12$ .
3. For  $2 \leq t < 4$ , we have  $\frac{dy(t)}{dt} = 0$ . Integrating both sides, we get  $y(t) = C_3$ .  
From the initial condition,  $y(2) = 12$ , we have  $C_3 = 12$  and thus we have  $y(4) = 12$ .
4. For  $4 \leq t < 6$ , we have  $\frac{dy(t)}{dt} = -1$ . Integrating both sides, we get  $y(t) = -t + C_4$ .  
From the initial condition,  $y(4) = 12$ , we have  $C_4 = 16$  and thus we have  $y(6) = 10$ .
5. For  $6 \leq t$ , we have  $\frac{dy(t)}{dt} = 0$ . Integrating both sides, we get  $y(t) = C_5$ .  
From the initial condition,  $y(6) = 10$ , we have  $C_5 = 10$ .
6. From the above we can finally write

$$f(t) = \begin{cases} 10, & t < 1 \\ 2t + 8, & 1 \leq t < 2 \\ 12, & 2 \leq t < 4 \\ -t + 16, & 4 \leq t < 6 \\ 10, & 6 \leq t \end{cases}$$


---



**Figure 181:** Problem 7.5, part (d).

**7.6.** Consider the differential equation

$$\dot{y} + 20y = 3 \cos(2t)u(t)$$

with the initial condition  $y(0) = 3$ .

- Find  $y_c(t)$ .
- Find  $y_p(t)$ .
- Find the total solution  $y(t)$ .
- Plot all three solutions  $v_c(t)$ ,  $v_p(t)$ ,  $v(t)$  and the driving force. When plotting the complementary solution, use the constant you derived in part (c).

**Solution.** Given

$$\frac{dy}{dt} + 20y = 3 \cos(2t)u(t) \quad (275)$$

$$y(0) = 3. \quad (276)$$

- We obtain the complementary solution  $y_c(t)$  by setting the driving force to zero, i.e., by solving

$$\frac{dy_c}{dt} + 20y_c = 0$$

Integrating this equation, we get

$$\begin{aligned}\int \frac{dy_c}{y_c} &= \int -20dt &\Rightarrow \\ \ln y_c &= -20t + c &\Rightarrow \\ y_c(t) &= Ce^{-20t}u(t)\end{aligned}\tag{277}$$

Note that, for the purposes of the homework, you did not have to derive equation 277; you could have used equation (7.35), page 341 in the textbook, directly.

- (b) Guessing the particular solution  $y_p(t)$  to be similar to the driving force, we set

$$\begin{aligned}y_p(t) &= B \cos(2t + \phi)u(t) \\ \frac{dy_p}{dt} &= -2B \sin(2t + \phi)\end{aligned}$$

Substituting in equation 275 (and using Equation 1.30, page 30, from the textbook), we get

$$\begin{aligned}-2B \sin(2t + \phi) + 20B \cos(2t + \phi) &= 3 \cos(2t) \\ \sqrt{(-2B)^2 + (20B)^2} \cos(2t + \phi - \arctan(\frac{-2}{20})) &= 3 \cos(2t)\end{aligned}$$

Comparing the amplitude and phase, we get

$$\begin{aligned}B &= \frac{3}{\sqrt{2^2 + 20^2}} = 0.1493 \\ \phi &= \arctan(\frac{-2}{20}) = -0.0997 &\Rightarrow \\ y_p(t) &= 0.1493 \cos(2t - 0.0997)u(t)\end{aligned}\tag{278}$$

- (c) The total solution  $y(t)$  can then be written as

$$\begin{aligned}y(t) &= y_c(t) + y_p(t) \\ &= Ce^{-20t}u(t) + 0.1493 \cos(2t - 0.0997)u(t).\end{aligned}$$

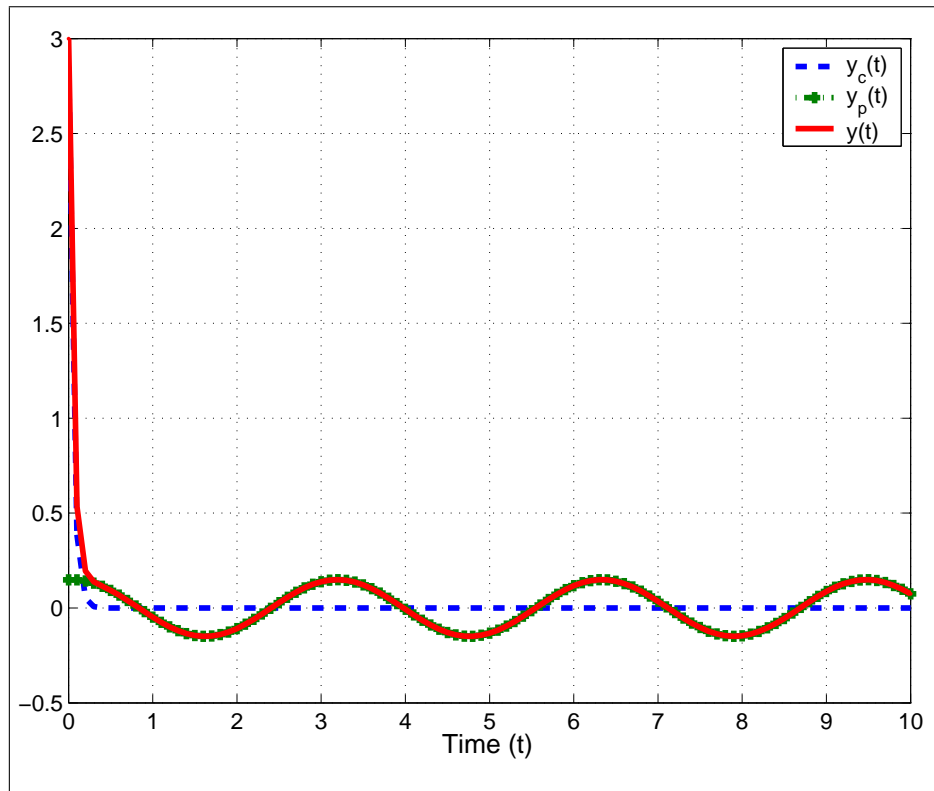
From initial conditions 276, we get

$$\begin{aligned} 3 &= Ce^{-20 \cdot 0} + 0.1493 \cos(2 \cdot 0 - 0.0997) \\ &= C + 0.1493 \cos(-0.0997) \quad \Rightarrow \\ C &= 2.8514 \end{aligned}$$

Thus the total solution is

$$y(t) = 2.8514e^{-20t}u(t) + 0.1493 \cos(2t - 0.0997)u(t).$$

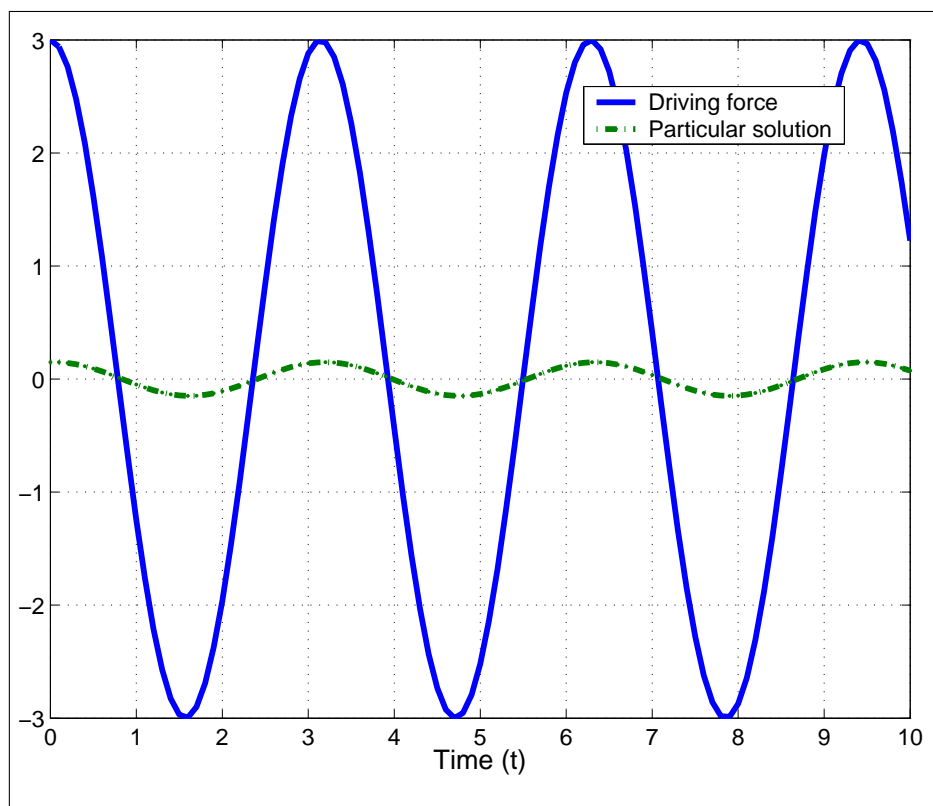
- (d) The matlab script titled `problem7_6.m` produces the graphs shown in figures 182 and 183.



**Figure 182:** Problem 7.6, part (d).

———— Different solutions of part (b) —————

- (b) Since the driving force  $y_s(t) = 3 \cos(2t)u(t)$  is in the form of sinusoidal function, we can apply the method in Section 7.3.2.2 for finding the particular solution  $y_p(t)$ . We have  $a = 20, A = 3, \omega_0 = 2$ . So, we guess



**Figure 183:** Problem 7.6, part (d).

$y_p(t) = B \sin(2t + \phi)u(t)$  (i.e., we guess a sine instead of a cosine function. Will it work?)

For all  $t > 0$ , we have

$$\frac{dy_p(t)}{dt} = 2B \cos(2t + \phi)$$

Substituting into the original equation, we get

$$2B \cos(2t + \phi) + 20B \sin(2t + \phi) = 3 \cos(2t)$$

From equation (1.30) in the textbook, we can rewrite the left hand side as

$$\begin{aligned} & B\sqrt{2^2 + 20^2} \cos(2t + \phi - \arctan(20/2)) \\ = & 2B\sqrt{101} \cos(2t + \phi - \arctan 10) \end{aligned}$$

thus,

$$2B\sqrt{101} = 3$$

and

$$\phi - \arctan 10 = 0$$

So we have

$$\begin{aligned} y_p(t) &= \frac{3}{2\sqrt{101}} \sin(2t + \arctan 10)u(t) \\ &\approx 0.1493 \sin(2t + 1.471) \end{aligned} \quad (279)$$

Is Equation 279 the same as equation 278? Can you reconcile the two?

Moreover, we may guess the solution to be in the form of  $y_p(t) = (B_1 \sin(2t) + B_2 \cos(2t))u(t)$ .

For all  $t > 0$ , we have

$$\frac{dy_p(t)}{dt} = 2B_1 \cos(2t) - 2B_2 \sin(2t)$$

Substituting into the original equation, we get

$$2B_1 \cos(2t) - 2B_2 \sin(2t) + 20B_1 \sin(2t) + 20B_2 \cos(2t) = 3 \cos(2t)$$

Equating the coefficients of the  $\sin(2t)$  terms (respectively the  $\cos(2t)$  terms), we get

$$\begin{cases} 20B_1 - 2B_2 = 0 \\ 2B_1 + 20B_2 = 3 \end{cases}$$

Solving the equations, we get

$$\begin{cases} B_1 = 0.01485 \\ B_2 = 0.1485 \end{cases}$$

$$\begin{aligned} y_p(t) &= (.01485 \sin(2t) + 0.1485 \cos(2t))u(t) \\ &\approx 0.1493 \sin(2t + 1.471) \end{aligned}$$

**7.12.** Repeat the previous problem for  $v_s(t) = 10e^{-0.01t}u(t)$ .

**Solution.** The DE is now

$$\frac{d^2v(t)}{dt^2} + 10\frac{dv(t)}{dt} + \frac{1}{2}v(t) = 5e^{-0.01t}u(t)$$

- (a) See equation 284.  
 (b) In order to find  $v_p(t)$ , we set  $\frac{1}{LC}v_s(t) = 5e^{-0.01t}u(t)$  and follow the procedure in section 7.4.2.3. Now  $A = 5, d = -0.01$ ; we can guess

$$v_p(t) = Ce^{-0.01t}u(t), C \neq 0.$$

For  $t > 0$ , we have

$$\begin{aligned}\frac{dv_p(t)}{dt} &= -0.01Ce^{-0.01t} \\ \frac{d^2v_p(t)}{dt^2} &= 10^{-4} \cdot Ce^{-0.01t}\end{aligned}$$

Plug into the original equation; we have

$$10^{-4} \cdot Ce^{-0.01t} + 10 \cdot -0.01Ce^{-0.01t} + \frac{1}{2}Ce^{-0.01t} = 5e^{-0.01t}$$

from which

$$C \approx 12.497$$

Therefore,

$$v_p(t) = 12.497e^{-0.01t}u(t)$$

- (c) The total solution is given by

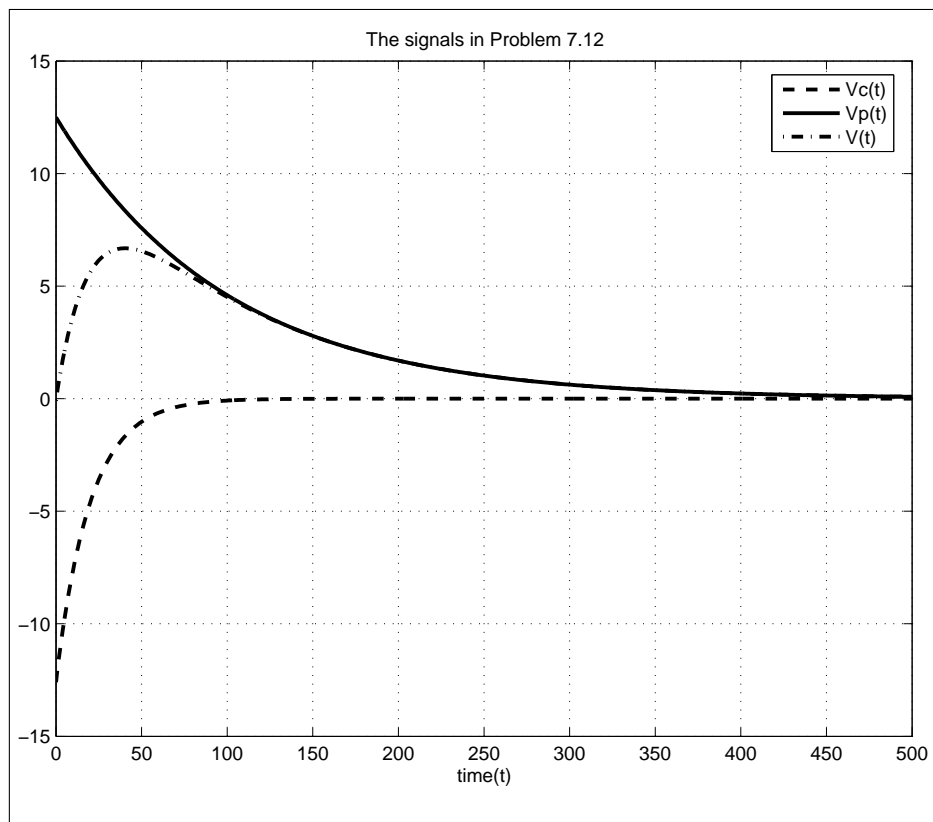
$$\begin{aligned}v(t) &= v_c(t) + v_p(t) \\ &= C_1e^{-.05t}u(t) + C_2e^{-9.95t}u(t) + 12.497e^{-0.01t}u(t)\end{aligned}$$

The system for the unknown constants is:

$$\begin{cases} C_1 + C_2 + 12.497 = 0 \\ (-5 + \frac{7}{2}\sqrt{2})C_1 + (-5 - \frac{7}{2}\sqrt{2})C_2 - 0.12497 = 1 \end{cases}$$

from which we get

$$\begin{aligned}C_1 &\approx -12.5 \\ C_2 &\approx -0.04\end{aligned}$$



**Figure 192:** Problem 7.12.

- (d) The matlab program titled *problem7\_12.m* was used. See Figure 192 for the plots.
- (e) The system is overdamped.

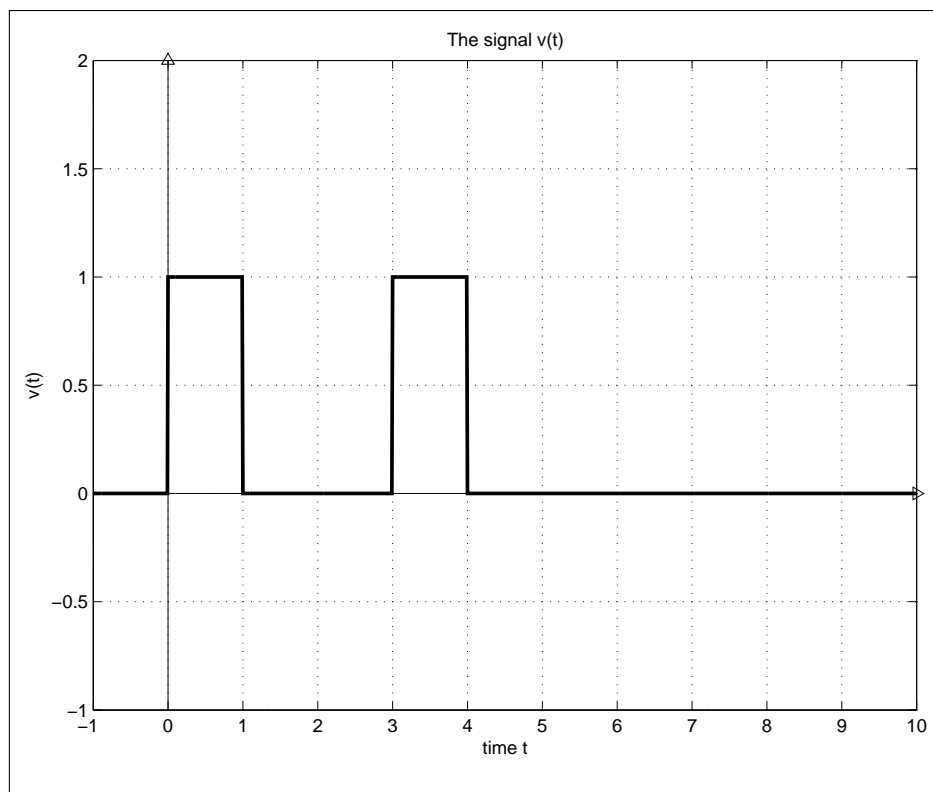
**8.3.** Find the Laplace transform of the signal  $v(t)$  in Figure 230, using Equation 8.5 in Example 8.3, and the linearity and time-shifting properties.

**Solution.** The signal  $v(t)$  can be decomposed into two pulses of unit duration starting at time 0 and 3 respectively. Thus

$$v(t) = (u(t) - u(t - 1)) + (u(t - 3) - u(t - 4))$$

Note that  $v(t)$  is a linear combination of time shifted unit step functions.

$$\begin{aligned}\mathcal{L}(u(t)) &= \frac{1}{s} \\ \mathcal{L}(u(t - t_0)) &= e^{-st_0} \frac{1}{s} \quad \Rightarrow \\ V(s) &= \frac{1}{s} - \frac{e^{-s}}{s} + \frac{e^{-3s}}{s} - \frac{e^{-4s}}{s}\end{aligned}$$



**Figure 230:** The signal  $v(t)$  in problem 8.3.

**8.10.** Use table 1 in page 404 and the linearity property to find  $v(t)$  for the following Laplace transforms.

$$(a) V(s) = \frac{-4}{s} + \frac{1-j2}{s+3+j4} + \frac{1+j2}{s+3-j4}$$

$$(b) V(s) = \frac{20e^{j\pi/4}}{s+2e^{-j\pi/4}} + \frac{20e^{-j\pi/4}}{s+2e^{j\pi/4}}$$

$$(c) V(s) = \frac{s-4}{s^2+25}$$

$$(d) V(s) = \frac{3}{s} + \frac{5-5j}{s+2-2j} + \frac{5+5j}{s+2+2j}$$

**Solution.**

(a) We have

$$\begin{aligned} V(s) &= \frac{-4}{s} + \frac{1-j2}{s+3+j4} + \frac{1+j2}{s+3-j4} \\ &= \frac{-4}{s} + \frac{(1-j2)(s+3-j4) + (1+j2)(s+3+j4)}{(s+3)^2+16} \\ &= \frac{-4}{s} + \frac{2(s-5)}{(s+3)^2+16} \\ &= \frac{-4}{s} + \frac{2(s+3)}{(s+3)^2+4^2} - \frac{16}{(s+3)^2+4^2} \end{aligned}$$

set  $V_1(s) = \frac{-4}{s}$ ,  $V_2(s) = 2 \cdot \frac{(s+3)}{(s+3)^2+16}$ ,  $V_3(s) = -\frac{16}{(s+3)^2+16} = -4 \cdot \frac{4}{(s+3)^2+16}$   
From table 1 on p.282, we find

$$\begin{aligned} v_1(t) &= -4u(t) \\ v_2(t) &= 2e^{-3t} \cos(4t)u(t) \\ v_3(t) &= -4e^{-3t} \sin(4t)u(t) \end{aligned}$$

From the linearity property,

$$v(t) = 2e^{-3t} (\cos(4t) - 2 \sin(4t)) u(t) - 4u(t)$$

(b) We have

$$\begin{aligned} V(s) &= \frac{20e^{\frac{j\pi}{4}}}{s+2e^{-\frac{j\pi}{4}}} + \frac{20e^{-\frac{j\pi}{4}}}{s+2e^{\frac{j\pi}{4}}} \\ &= \frac{20(e^{\frac{j\pi}{4}}s + 2e^{\frac{j\pi}{2}} + e^{-\frac{j\pi}{4}}s + 2e^{-\frac{j\pi}{2}})}{s^2 + 2(e^{-\frac{j\pi}{4}} + e^{\frac{j\pi}{4}})s + 4} \\ &= \frac{20(2\cos(\frac{\pi}{4})s + 2 \cdot 2\cos(\frac{\pi}{2}))}{s^2 + 2(2\cos(\frac{\pi}{4}))s + 4} \end{aligned}$$

$$\begin{aligned}
&= \frac{20(2\frac{\sqrt{2}}{2}s)}{s^2 + 4\frac{\sqrt{2}}{2}s + 4} \\
&= \frac{20\sqrt{2}s}{s^2 + 2\sqrt{2}s + 2 + 2} \\
&= 20\sqrt{2} \frac{s}{(s + \sqrt{2})^2 + \sqrt{2}^2} \\
&= 20\sqrt{2} \left[ \frac{s + \sqrt{2}}{(s + \sqrt{2})^2 + \sqrt{2}^2} - \frac{\sqrt{2}}{(s + \sqrt{2})^2 + \sqrt{2}^2} \right]
\end{aligned}$$

set  $V_1(s) = 20\sqrt{2} \frac{s+\sqrt{2}}{(s+\sqrt{2})^2+\sqrt{2}^2}$ ,  $V_2(s) = -20\sqrt{2} \frac{\sqrt{2}}{(s+\sqrt{2})^2+\sqrt{2}^2}$

From table 1 on p.282, we find

$$\begin{aligned}
v_1(t) &= 20\sqrt{2}e^{-\sqrt{2}t} \cos(\sqrt{2}t)u(t) \\
v_2(t) &= -20\sqrt{2}e^{-\sqrt{2}t} \sin(\sqrt{2}t)u(t)
\end{aligned}$$

From the linearity property,

$$v(t) = 20\sqrt{2}e^{-\sqrt{2}t}(\cos(\sqrt{2}t) - \sin(\sqrt{2}t))u(t)$$

(c) We have

$$\begin{aligned}
V(s) &= \frac{s-4}{s^2+25} \\
&= \frac{s}{s^2+5^2} - \frac{4}{5} \cdot \frac{5}{s^2+5^2}
\end{aligned}$$

set  $V_1(s) = \frac{s}{s^2+5^2}$ ,  $V_2(s) = -\frac{4}{5} \cdot \frac{5}{s^2+5^2}$

From table 1 on p.282, we find

$$\begin{aligned}
v_1(t) &= \cos(5t)u(t) \\
v_2(t) &= -\frac{4}{5} \sin(5t)u(t)
\end{aligned}$$

From the linearity property,

$$v(t) = \cos(5t)u(t) - \frac{4}{5} \sin(5t)u(t)$$

(d) We have

$$V(s) = \frac{3}{s} + \frac{5-5j}{s+2-2j} + \frac{5+5j}{s+2+2j}$$

$$\begin{aligned}
&= \frac{3}{s} + \frac{(5-5j)(s+2+2j) + (5+5j)(s+2-2j)}{(s+2)^2 + 4} \\
&= \frac{3}{s} + \frac{10(s+4)}{(s+2)^2 + 2^2} \\
&= \frac{3}{s} + \frac{10(s+2)}{(s+2)^2 + 2^2} + \frac{10 \cdot 2}{(s+2)^2 + 2^2}
\end{aligned}$$

$$V_1(s) = \frac{3}{s}, V_2(s) = \frac{10(s+2)}{(s+2)^2 + 2^2}, V_3(s) = \frac{10 \cdot 2}{(s+2)^2 + 2^2}$$

From table 1 on p.282, we find

$$\begin{aligned}
v_1(t) &= 3u(t) \\
v_2(t) &= 10e^{-2t} \cos(2t)u(t) \\
v_3(t) &= 10e^{-2t} \sin(2t)u(t)
\end{aligned}$$

From the linearity property,

$$v(t) = 10e^{-2t} (\cos(2t) + \sin(2t)) u(t) + 3u(t)$$


---

**8.12.** Find the partial fraction expansion of

$$V(s) = \frac{8s^2 + 20s + 16}{s(s^2 + 4s + 8)}$$

by hand. Verify your results by using the Matlab function *residue*.

**Solution.** We have

$$V(s) = \frac{8s^2 + 20s + 16}{s(s^2 + 4s + 8)}$$

The roots for  $(s^2 + 4s + 8)$  are  $-2 \pm 2j$   
rewrite  $V(s)$  as

$$V(s) = \frac{K_1}{s} + \frac{K_2}{s - (-2 + 2j)} + \frac{K_2^*}{s - (-2 - 2j)}$$

Now

$$\begin{aligned} K_1 &= sV(s)|_{s=0} \\ &= \frac{8s^2 + 20s + 16}{(s^2 + 4s + 8)}|_{s=0} \\ &= \frac{16}{8} = 2 \end{aligned}$$

$$\begin{aligned} K_2 &= (s - (-2 + 2j))V(s)|_{s=-2+2j} \\ &= \frac{8s^2 + 20s + 16}{s(s - (-2 - 2j))}|_{s=-2+2j} \\ &= \frac{8(-2 + 2j)^2 + 20(-2 + 2j) + 16}{(-2 + 2j)(-2 + 2j + 2 + 2j)} \\ &= \frac{8(4 - 8j - 4) + 40j - 24}{-8 - 8j} \\ &= \frac{-24 - 24j}{-8 - 8j} = 3 \end{aligned}$$

$$K_2^* = 3$$

Therefore,

$$\begin{aligned} V(s) &= \frac{2}{s} + \frac{3}{s - (-2 + 2j)} + \frac{3}{s - (-2 - 2j)} \\ &= \frac{2}{s} + \frac{6s + 12}{4s + s^2 + 8} \end{aligned}$$

The matlab code

```

% Problem 8.12

clear
Num = [8 20 16];
Denom = [ 1 4 8 0] ;
[Residues, Poles, K] = residue(Num, Denom)

returns

Residues =

    3.0000 - 0.0000i
    3.0000 + 0.0000i
    2.0000

Poles =

   -2.0000 + 2.0000i
   -2.0000 - 2.0000i
         0

K =

[]

```

which matches our solution.

---

**8.21.** The transfer function of a system is given by

$$H(s) = \frac{1}{s^2 + 2s + 10}$$

The Laplace transform of the driving force is equal to  $1/s$ . Determine the Laplace transform of the output; then determine the output in the time domain.

**Solution.** We know that  $V(s) = H(s) \cdot V_s(s)$ ; since  $V_s(s) = 1/s$ , we can write

$$V(s) = \frac{1}{s(s^2 + 2s + 10)}$$

The roots for  $(s^2 + 2s + 10)$  are  $s = 1 + 3j$  and  $s = 1 - 3j$

Rewrite

$$V(s) = \frac{K_1}{s} + \frac{K_2}{s - (-1 + 3j)} + \frac{K_2^*}{s - (-1 - 3j)}$$

Now

$$\begin{aligned} K_1 &= (s)V(s)|_{s=0} \\ &= \left. \frac{1}{s^2 + 2s + 10} \right|_{s=0} \\ &= \frac{1}{10} \end{aligned}$$

$$\begin{aligned} K_2 &= (s - (-1 + 3j))Y(s)|_{s=-1+3j} \\ &= \left. \frac{1}{s(s - (-1 - 3j))} \right|_{s=-1+3j} \\ &= \frac{1}{100}(-4 + 3j) \end{aligned}$$

$$K_2^* = \frac{1}{100}(-4 - 3j)$$

so

$$V(s) = \frac{1}{10} \frac{1}{s} + \frac{-4 + 3j}{100} \frac{1}{s - (-1 + 3j)} + \frac{-4 - 3j}{100} \frac{1}{s - (-1 - 3j)}$$

Solving for  $v(t)$ , we get

$$v(t) = \left( \frac{1}{10} + \frac{1}{10} e^{-t} \cos(3t - 0.6435) \right) u(t)$$

**\*9.2.** Consider a periodic function  $f(t)$  with period  $T$ . Let  $\omega_0 = 2\pi/T$ . Let  $g_n(t) = f(t) \cos(n\omega_0 t)$ . Find the period  $T_n$  of the function  $g_n(t)$ . Determine the relationship between  $T_n$  and  $T$ .

**Solution.** For any (positive) integer  $n$ , note that the period of the signal  $\cos(n\omega_0 t) = \cos(2\pi \frac{n}{T} t)$  is  $T/n$ , so  $T$  is an integer multiple of  $T/n$ . We have

$$\begin{aligned} g_n(t+T) &= f(t+T) \cos(n\omega_0(t+T)) \\ &= f(t) \cos(n\omega_0 t + 2\pi n) = f(t) \cos(n\omega_0 t) \\ &= g_n(t) \end{aligned}$$

and thus  $T$  is a period. Observe that  $T$  is also the smallest positive number that satisfies  $g_n(t+T) = g_n(t)$  (due to the presence of the  $f(t)$  term), so *the* period of  $g_n(t)$  is  $T_n = T$ .

---

**9.11.** Without using any integrals, determine the Fourier series coefficients,  $a_k$ ,  $b_k$  and  $\alpha_k$ , of the following functions. Note the frequency that corresponds to each index  $k$ . The time variable is measured in seconds. You will need to determine the period,  $T$ , over which to determine the coefficients. *Use the shortest possible period.*

- (a)  $s_1(t) = 0.45 \cos(7.1t)$
- (b)  $s_2(t) = 3 \cos(4.3t) + 2 \sin(12.9t)$
- (c)  $s_3(t) = 3.0 \cos(2\pi 30t) - 1.7 \cos(2\pi 100t)$
- (d)  $s_4(t) = 3.0 \cos(2\pi 30t) - 1.7 \sin(2\pi 30t)$
- (e)  $s_5(t) = 20 \cos(2\pi 30t) - 17 \cos(2\pi 100t - 5\pi/6) + 8 \cos(2\pi 200t + \pi/4)$

**Solution.**

(a) We have

$$\begin{aligned} s_1(t) &= .45 \cos(7.1t) \\ &= .45 \cos \left[ \left( \frac{2\pi}{2\pi/7.1} \right) t \right] \end{aligned}$$

Therefore,  $T = \frac{2\pi}{7.1}$  seconds, and

$$a_k = \begin{cases} .45 & , k = 1 \\ 0 & , \text{o.w.} \end{cases}$$

$$b_k = 0, k = 1, 2, \dots$$

from Equations (9.7)-(9.9), we have

$$\alpha_k = \begin{cases} \frac{.45}{2} & , k = \pm 1 \\ 0 & , \text{o.w.} \end{cases}$$

(b)

$$\begin{aligned} s_2(t) &= 3 \cos(4.3t) + 2 \sin(12.9t) \\ &= 3 \cos\left(\frac{2\pi}{2\pi/4.3}t\right) + 2 \sin\left(\frac{2\pi}{2\pi/(3 \cdot 4.3)}t\right) \\ &= 3 \cos\left(\frac{2\pi \cdot 1}{2\pi/4.3}t\right) + 2 \sin\left(\frac{2\pi \cdot 3}{2\pi/4.3}t\right) \end{aligned}$$

therefore,  $T = \frac{2\pi}{4.3} = 1.4612$  seconds, and

$$a_k = \begin{cases} 3 & , k = 1 \\ 0 & , \text{o.w.} \end{cases}$$

$$b_k = \begin{cases} 2 & , k = 3 \\ 0 & , \text{o.w.} \end{cases}$$

from Equations (9.7)-(9.9), we have

$$\alpha_k = \begin{cases} \frac{3}{2} & , k = \pm 1 \\ -\frac{2j}{2} = -j & , k = 3 \\ j & , k = -3 \\ 0 & , \text{o.w.} \end{cases}$$

(c) We have

$$\begin{aligned} s_3(t) &= 3.0 \cos(2\pi 30t) - 1.7 \cos(2\pi 100t) \\ &= 3.0 \cos\left(\frac{2\pi \cdot 3}{1/10}t\right) - 1.7 \cos\left(\frac{2\pi \cdot 10}{1/10}t\right) \end{aligned}$$

therefore,  $T = \frac{1}{10}$  seconds, and

$$a_k = \begin{cases} 3.0 & , k = 3 \\ -1.7 & , k = 10 \\ 0 & , \text{o.w.} \end{cases}$$

$$b_k = 0, k = 1, 2, \dots$$

from equations (9.7)-(9.9), we have

$$\alpha_k = \begin{cases} \frac{3}{2} & , k = \pm 3 \\ -\frac{1.7}{2} & , k = \pm 10 \\ 0 & , \text{o.w.} \end{cases}$$

(d) We have

$$\begin{aligned} s_4(t) &= 3.0 \cos(2\pi 30t) - 1.7 \sin(2\pi 30t) \\ &= 3.0 \cos\left(\frac{2\pi \cdot 1}{1/30}t\right) - 1.7 \sin\left(\frac{2\pi \cdot 1}{1/30}t\right) \end{aligned}$$

therefore,  $T = \frac{1}{30}$  seconds, and

$$a_k = \begin{cases} 3 & , k = 1 \\ 0 & , \text{o.w.} \end{cases}$$

$$b_k = \begin{cases} -1.7 & , k = 1 \\ 0 & , \text{o.w.} \end{cases}$$

from Equations (9.7)-(9.9), we have

$$\alpha_k = \begin{cases} \frac{3+1.7j}{2} & , k = 1 \\ \frac{3-1.7j}{2} & , k = -1 \\ 0 & , \text{o.w.} \end{cases}$$

(e) We have

$$\begin{aligned} s_5(t) &= 20 \cos(2\pi 30t) - 17 \cos(2\pi 100t - \frac{5}{6}\pi) + 8 \cos(2\pi 200t + \frac{\pi}{4}) \\ &= 20 \cos(2\pi 30t) - 17 \cos(2\pi 100t) \cdot \cos(\frac{5}{6}\pi) - 17 \sin(2\pi 100t) \sin(\frac{5}{6}\pi) \\ &\quad + 8 \cos(2\pi 200t) \cdot \cos(\frac{\pi}{4}) - 8 \sin(2\pi 200t) \cdot \sin(\frac{\pi}{4}) \\ &= 20 \cos(2\pi 30t) + \frac{17\sqrt{3}}{2} \cos(2\pi 100t) - \frac{17}{2} \sin(2\pi 100t) \\ &\quad + 4\sqrt{2} \cos(2\pi 200t) - 4\sqrt{2} \sin(2\pi 200t) \\ &= 20 \cos(\frac{2\pi \cdot 3}{1/10}t) + \frac{17\sqrt{3}}{2} \cos(\frac{2\pi \cdot 10}{1/10}t) \\ &\quad - \frac{17}{2} \sin(\frac{2\pi \cdot 10}{1/10}t) + 4\sqrt{2} \cos(\frac{2\pi \cdot 20}{1/10}t) - 4\sqrt{2} \sin(\frac{2\pi \cdot 20}{1/10}t) \end{aligned}$$

therefore,  $T = \frac{1}{10}$  seconds, and

$$a_k = \begin{cases} 20 & , k = 3 \\ \frac{17}{2}\sqrt{3} & , k = 10 \\ 4\sqrt{2} & , k = 20 \\ 0 & , \text{o.w.} \end{cases}$$

$$b_k = \begin{cases} -\frac{17}{2} & , k = 10 \\ -4\sqrt{2} & , k = 20 \\ 0 & , \text{o.w.} \end{cases}$$

from Equations (9.7)-(9.9), we have

$$\alpha_k = \begin{cases} 10 & , k = \pm 3 \\ \frac{17\sqrt{3}+17j}{4} & , k = 10 \\ \frac{17\sqrt{3}-17j}{4} & , k = -10 \\ 2\sqrt{2} + 2\sqrt{2}j & , k = 20 \\ 2\sqrt{2} - 2\sqrt{2}j & , k = -20 \\ 0 & , o.w. \end{cases}$$


---

**9.12.** One period of  $s(t)$  is defined by

$$s(t) = \begin{cases} 2, & 0 \leq t \leq 1 \\ -2, & 1 < t \leq 3 \end{cases}$$

Use direct integration to find the  $\alpha_n$  coefficients.

**Solution.** We have  $T = 3, \omega_0 = \frac{2\pi}{3}$ . Then, from Equation (9.14), we have

$$\begin{aligned} \alpha_n &= \frac{1}{3} \int_0^3 s(t) e^{-jn \frac{2\pi}{3} t} dt \\ &= \frac{1}{3} \left( \int_0^1 2e^{-jn \frac{2\pi}{3} t} dt + \int_1^3 -2e^{-jn \frac{2\pi}{3} t} dt \right) \\ &= \frac{1}{3} \left( \left. \frac{2e^{-jn \frac{2\pi}{3} t}}{-jn \frac{2\pi}{3}} \right|_0^1 + \left. \frac{-2e^{-jn \frac{2\pi}{3} t}}{-jn \frac{2\pi}{3}} \right|_1^3 \right) \\ &= \frac{1}{3} \frac{-3}{j2\pi n} 2 \left( e^{-jn \frac{2\pi}{3} t} \Big|_0^1 - e^{-jn \frac{2\pi}{3} t} \Big|_1^3 \right) \\ &= \frac{-1}{j\pi n} \left( \left( e^{-jn \frac{2\pi}{3}} - 1 \right) - \left( e^{-j2\pi n} - e^{-jn \frac{2\pi}{3}} \right) \right) \end{aligned}$$

Since  $e^{-j2\pi n} = 1$  for  $n = \dots, -1, 0, 1, \dots$ , we can write

$$\begin{aligned} \alpha_n &= \frac{-1}{j\pi n} \left( e^{-jn \frac{2\pi}{3}} - 1 - 1 + e^{-jn \frac{2\pi}{3}} \right) \\ &= \frac{-2}{j\pi n} \left( e^{-jn \frac{2\pi}{3}} - 1 \right) \\ &= \frac{-2}{j\pi n} e^{-jn \frac{\pi}{3}} \left( e^{-jn \frac{\pi}{3}} - e^{jn \frac{\pi}{3}} \right) \\ &= \frac{2}{j\pi n} e^{-jn \frac{\pi}{3}} 2j \sin \left( \frac{\pi n}{3} \right) \\ &= \frac{4 \sin \left( \frac{\pi n}{3} \right) e^{-\frac{j\pi n}{3}}}{\pi n} \\ &= \frac{4 \sin \left( \frac{\pi n}{3} \right)}{\pi n} \left( \cos \left( \frac{\pi n}{3} \right) - j \sin \left( \frac{\pi n}{3} \right) \right) \end{aligned}$$

Moreover,

$$\begin{aligned} a_n &= 2\operatorname{Re}(\alpha_n) \\ &= 2 * \frac{4 \sin \left( \frac{\pi n}{3} \right) \cos \left( \frac{\pi n}{3} \right)}{\pi n} \\ &= \frac{4 \sin \left( \frac{2\pi n}{3} \right)}{\pi n} \end{aligned}$$

$$\begin{aligned}
b_n &= -2\operatorname{Im}(\alpha_n) \\
&= -2 * \frac{4 \sin(\frac{\pi n}{3}) * -\sin(\frac{\pi n}{3})}{\pi n} \\
&= \frac{4 [1 - \cos(\frac{2\pi n}{3})]}{\pi n}
\end{aligned}$$


---

**9.15.** Let  $T = 10$ . One period of  $s(t)$  is defined by

$$s(t) = \begin{cases} 0, & -T/2 \leq t < 0 \\ 2, & 0 \leq t \leq 1 \\ -1, & 1 < t \leq 2 \\ 0, & 2 < t < T/2 \end{cases}$$

Use direct integration to find the  $\alpha_n$  coefficients.

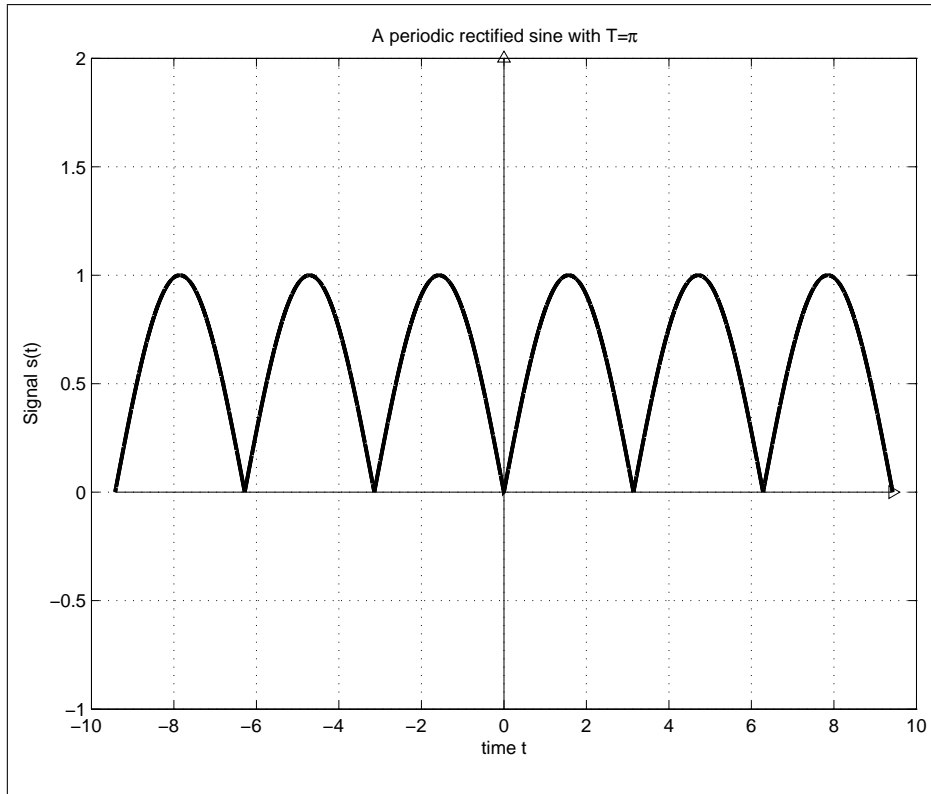
**Solution.** We have  $T = 10, \omega_0 = \frac{\pi}{5}$ . Then,

$$\begin{aligned} \alpha_n &= \frac{1}{10} \left( \int_{-5}^5 s(t) e^{-jn \frac{\pi}{5} t} dt \right) \\ &= \frac{1}{10} \left( \int_{-5}^0 0 e^{-jn \frac{\pi}{5} t} dt + \int_0^1 2 e^{-jn \frac{\pi}{5} t} dt + \int_1^2 -1 e^{-jn \frac{\pi}{5} t} dt + \int_2^5 0 e^{-jn \frac{\pi}{5} t} dt \right) \\ &= \frac{1}{10} \left( 0 + \left. \frac{2e^{-j \frac{\pi n t}{5}}}{\frac{-j \pi n}{5}} \right|_0^1 - \left. \frac{e^{-j \frac{\pi n t}{5}}}{\frac{-j \pi n}{5}} \right|_1^2 + 0 \right) \\ &= \frac{-1}{j 2 \pi n} \left( 2 \left( e^{-j \frac{\pi n}{5}} - 1 \right) - \left( e^{-j \frac{2 \pi n}{5}} - e^{-j \frac{\pi n}{5}} \right) \right) \\ &= \frac{-1}{j 2 \pi n} \left( 2 e^{-j \frac{\pi n}{10}} \left( e^{-j \frac{\pi n}{10}} - e^{j \frac{\pi n}{10}} \right) - e^{-j \frac{3 \pi n}{10}} \left( e^{-j \frac{\pi n}{10}} - e^{j \frac{\pi n}{10}} \right) \right) \\ &= \frac{-1}{j 2 \pi n} \left( 2 e^{-j \frac{\pi n}{10}} 2j \sin\left(\frac{-\pi n}{10}\right) - e^{-j \frac{3 \pi n}{10}} 2j \sin\left(\frac{-\pi n}{10}\right) \right) \\ &= \frac{1}{\pi n} \left( 2 e^{-j \frac{\pi n}{10}} \sin\left(\frac{\pi n}{10}\right) - e^{-j \frac{3 \pi n}{10}} \sin\left(\frac{\pi n}{10}\right) \right) \\ &= \frac{\sin\left(\frac{\pi n}{10}\right)}{\pi n} \left( 2 e^{-j \frac{\pi n}{10}} - e^{-j \frac{3 \pi n}{10}} \right) \end{aligned}$$


---

**9.17.** Find the Fourier series for the signal shown in Figure 251. The period is  $T = \pi$ . Mathematically, the signal can be defined as

$$s(t) = |\sin(t)|.$$



**Figure 251:** A periodic rectified sine.

**Solution.** We have  $T = \pi$ ,  $\omega_0 = 2$ . Then,

$$\begin{aligned} \alpha_n &= \frac{1}{\pi} \int_0^\pi \sin(t) e^{-j2nt} dt \\ &= \frac{1}{\pi} \int_0^\pi \frac{1}{2j} (e^{jt} - e^{-jt}) e^{-j2nt} dt \\ &= \frac{1}{j2\pi} \int_0^\pi (e^{j(1-2n)t} - e^{-j(1+2n)t}) dt \\ &= \frac{1}{j2\pi} \left( \frac{e^{j(1-2n)t}}{j(1-2n)} \Big|_0^\pi - \frac{e^{-j(1+2n)t}}{-j(1+2n)} \Big|_0^\pi \right) \end{aligned}$$

$$\begin{aligned}
&= \frac{1}{j2\pi} \left( \frac{-1-1}{j(1-2n)} + \frac{-1-1}{j(1+2n)} \right) \\
&= \frac{1}{\pi} \left( \frac{1}{1-2n} + \frac{1}{1+2n} \right) \\
&= \frac{1}{\pi} \cdot \frac{2}{1-4n^2}
\end{aligned}$$


---

**9.20.** Determine whether  $s(t)$  is an even or odd function.

- (a)  $s(t) = t \cos(2\pi 20t)$
- (b)  $s(t) = t \cos(2\pi 20t - \pi/2)$
- (c)  $s(t) = t \cos(2\pi 20t + \pi/2)$
- (d)  $s(t) = t \cos(2\pi 20t + \pi/4)$
- (e)  $s(t) = 1 + \sin(2\pi 60t)$

**Solution.**

(a)

$$\begin{aligned} s(t) &= t \cos(2\pi 20t) \\ s(-t) &= (-t) \cos(2\pi 20(-t)) \\ &= -t \cos(2\pi 20t) \\ &= -s(t) \end{aligned}$$

Therefore,  $s(t)$  is an odd function

(b)

$$\begin{aligned} s(t) &= t \cos(2\pi 20t - \frac{\pi}{2}) \\ &= t \sin(2\pi 20t) \\ s(-t) &= (-t) \sin(2\pi 20(-t)) \\ &= (-t) \cdot (-\sin(2\pi 20t)) \\ &= t \sin(2\pi 20t) \\ &= s(t) \end{aligned}$$

Therefore,  $s(t)$  is an even function

(c)

$$\begin{aligned} s(t) &= t \cos(2\pi 20t + \frac{\pi}{2}) \\ &= -t \sin(2\pi 20t) \\ s(-t) &= -(-t) \sin(2\pi 20(-t)) \\ &= t \cdot -\sin(2\pi 20t) \\ &= -t \sin(2\pi 20t) \\ &= s(t) \end{aligned}$$

Therefore,  $s(t)$  is an even function

(d)

$$\begin{aligned}s(t) &= t \cos\left(2\pi 20t + \frac{\pi}{4}\right) \\&= t \left[ \cos(2\pi 20t) \cos\left(\frac{\pi}{4}\right) - \sin(2\pi 20t) \sin\left(\frac{\pi}{4}\right) \right] \\&= t \left[ \frac{\sqrt{2}}{2} \cos(2\pi 20t) - \frac{\sqrt{2}}{2} \sin(2\pi 20t) \right] \\&= \frac{\sqrt{2}}{2} t [\cos(2\pi 20t) - \sin(2\pi 20t)] \\s(-t) &= \frac{\sqrt{2}}{2} \cdot (-t) \cdot [\cos(2\pi 20(-t)) - \sin(2\pi 20(-t))] \\&= \frac{\sqrt{2}}{2} \cdot (-t) \cdot [\cos(2\pi 20(t)) + \sin(2\pi 20(t))] \\&\neq s(t) \text{ or } -s(t)\end{aligned}$$

Therefore,  $s(t)$  is neither an odd function nor an even function

(e)

$$\begin{aligned}s(t) &= 1 + \sin(2\pi 60t) \\s(-t) &= 1 + \sin(2\pi 60(-t)) \\&= 1 - \sin(2\pi 60(-t)) \\&\neq s(t) \text{ or } -s(t)\end{aligned}$$

Therefore,  $s(t)$  is neither an odd function nor an even function

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**9.21.** One period of the function  $s(t)$  is defined on  $[0,4]$  as

$$s(t) = \begin{cases} 1, & 0 < t \leq 1 \\ 2, & 1 < t \leq 2 \\ 1, & 2 < t \leq 3 \\ 0, & 3 < t \leq 4 \end{cases}$$

Is the signal  $s(t)$  even, odd or neither?

**Solution.** Since the time period  $T = 4$ ,

$$s(t) = s(t - 4) = \begin{cases} 1, & -4 < t \leq -3 \\ 2, & -3 < t \leq -2 \\ 1, & -2 < t \leq -1 \\ 0, & -1 < t \leq 0 \end{cases}$$

By inspection, we can conclude that the signal is neither odd nor even. In general, the easiest way would be to compute the area under the curve. For an odd function  $f(t)$ ,

$$\int_{-T/2}^{T/2} f(t) dt = 0$$

For an even function  $f(t)$ , we can write

$$\int_{-T/2}^{T/2} f(t) dt = 2 \int_0^{T/2} f(t) dt$$

In this case,

$$\begin{aligned} \int_{-2}^2 s(t) dt &= \int_{-2}^{-1} 1 dt + \int_{-1}^0 0 dt + \int_0^1 1 dt + \int_1^2 2 dt \\ &= 1 + 0 + 1 + 2 = 4 \neq 0 \end{aligned}$$

$$\begin{aligned} \int_0^2 s(t) dt &= \int_0^1 1 dt + \int_1^2 2 dt \\ &= 1 + 2 = 3 \neq \frac{4}{2} \end{aligned}$$

This confirms that the signal belongs to neither category.

For those who are wondering what this has got to do with Fourier series, we have the more cerebral approach! Remember that the sine function is odd while the cosine function is even. Thus any odd signal upon decomposition

would contain only sin terms ( $a_n = 0$ ) and an even function only cosine terms ( $b_n = 0$ ).  $\omega_0 = \frac{2\pi}{T} = \frac{\pi}{2}$ .

$$\begin{aligned}
a_n &= \frac{2}{4} \int_{-2}^2 s(t) \cos(n \frac{\pi}{2} t) dt \\
&= \frac{2}{4} \left( \int_{-2}^{-1} 1 \cos(n \frac{\pi}{2} t) dt + \int_{-1}^0 0 \cos(n \frac{\pi}{2} t) dt + \int_0^1 1 \cos(n \frac{\pi}{2} t) dt + \int_1^2 2 \cos(n \frac{\pi}{2} t) dt \right) \\
&= \frac{1}{2} \left( \frac{\sin(\frac{\pi n t}{2})}{\pi n / 2} \Big|_{-2}^{-1} + 0 + \frac{\sin(\frac{\pi n t}{2})}{\pi n / 2} \Big|_0^1 + 2 \frac{\sin(\frac{\pi n t}{2})}{\pi n / 2} \Big|_1^2 \right) \\
&= \frac{1}{\pi n} \left( \left( \sin\left(\frac{-\pi n}{2}\right) - \sin(-\pi n) \right) + \left( \sin\left(\frac{\pi n}{2}\right) - \sin(0) \right) + 2 \left( \sin(\pi n) - \sin\left(\frac{\pi n}{2}\right) \right) \right) \\
\Rightarrow a_n &= \frac{-2 \sin(\frac{n\pi}{2})}{\pi n}
\end{aligned}$$

as  $\sin(n\pi) = 0 \quad \forall \quad n$ .

$$\begin{aligned}
b_n &= \frac{2}{4} \int_{-2}^2 s(t) \sin(n \frac{\pi}{2} t) dt \\
&= \frac{2}{4} \left( \int_{-2}^{-1} 1 \sin(n \frac{\pi}{2} t) dt + \int_{-1}^0 0 \sin(n \frac{\pi}{2} t) dt + \int_0^1 1 \sin(n \frac{\pi}{2} t) dt + \int_1^2 2 \sin(n \frac{\pi}{2} t) dt \right) \\
&= \frac{1}{2} \left( \frac{-\cos(\frac{\pi n t}{2})}{\pi n / 2} \Big|_{-2}^{-1} + 0 + \frac{-\cos(\frac{\pi n t}{2})}{\pi n / 2} \Big|_0^1 + 2 \frac{-\cos(\frac{\pi n t}{2})}{\pi n / 2} \Big|_1^2 \right) \\
&= \frac{-1}{\pi n} \left( \left( \cos\left(\frac{-\pi n}{2}\right) - \cos(-\pi n) \right) + \left( \cos\left(\frac{\pi n}{2}\right) - \cos(0) \right) + 2 \left( \cos(\pi n) - \cos\left(\frac{\pi n}{2}\right) \right) \right) \\
\Rightarrow b_n &= \frac{1 - \cos(n\pi)}{\pi n}
\end{aligned}$$

as  $\cos(\frac{n\pi}{2}) = 0 \quad \forall \quad n$ .

Since  $a_n \neq 0, b_n \neq 0$ , the signal is neither odd nor even.

**9.25.** The  $S(n)$  values for  $n = 1, 2, 3, 4$  are given by

$$\{3 + j, 0, -3 - 3j, 0\}.$$

Find the values of  $S(n)$  for  $n \in \{-1, -2, -3, -4\}$ .

**Solution.** We note that  $S(n)$  and  $S(-n)$  are conjugates of each other. Hence  $S(n)$  for  $n = -1, -2, -3, -4$  are given by  $\{3 - j, 0, -3 + 3j, 0\}$ .

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**9.30.** Find the fundamental frequency,  $\omega_0$ , and the coefficients of the Fourier series of the signal defined by

$$v_p(t) = 3 \cos(2\pi 720t - 4\pi/5) + 2 \cos(2\pi 240t + \pi/7)$$

Plot the magnitude and phase spectra as functions of frequency in Hertz.

**Solution.** We have

$$\begin{aligned} v_p(t) &= 3 \cos(2\pi 720t - 4\pi/5) + 2 \cos(2\pi 240t + \pi/7) \\ &= 3 \cos(2\pi 720t) \cos(4\pi/5) + 3 \sin(2\pi 720t) \sin(4\pi/5) \\ &+ 2 \cos(2\pi 240t) \cos(\pi/7) - 2 \sin(2\pi 240t) \sin(\pi/7) \\ &= -2.427 \cos\left(\frac{2\pi \cdot 3}{1/240}t\right) + 1.763 \sin\left(\frac{2\pi \cdot 3}{1/240}t\right) \\ &+ 1.8 \cos\left(\frac{2\pi \cdot 1}{1/240}t\right) - 0.867 \sin\left(\frac{2\pi \cdot 1}{1/240}t\right) \end{aligned}$$

therefore  $T = \frac{1}{240}$  sec And  $\omega_0 = \frac{2\pi}{T} = 480\pi$

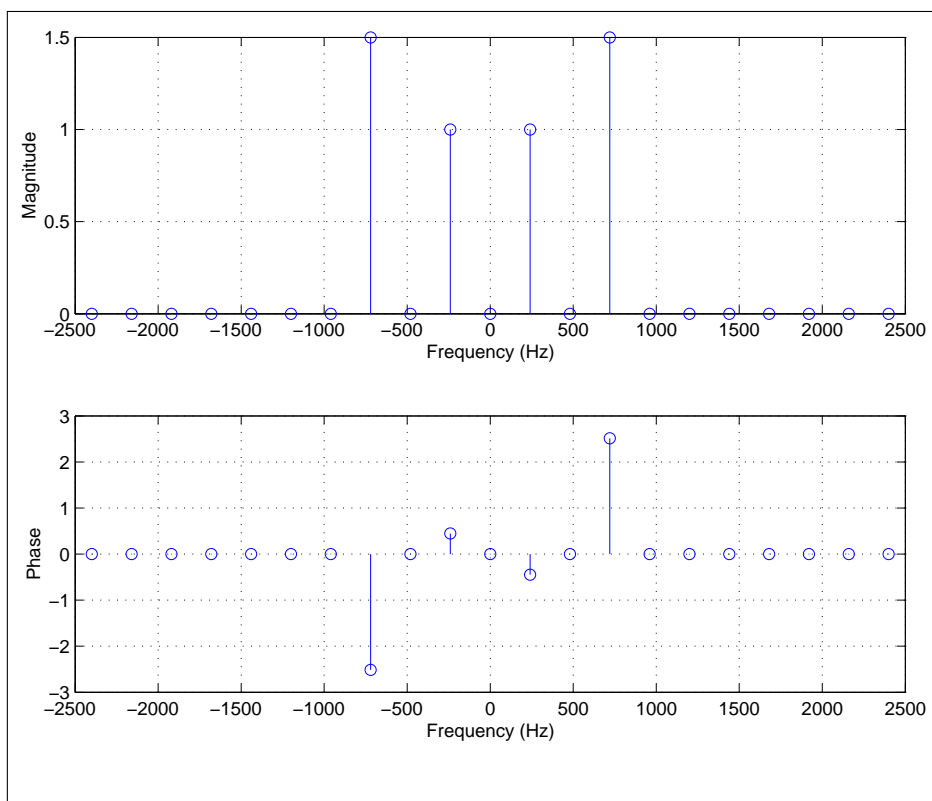
$$a_k = \begin{cases} 1.8, & k = 1 \\ -2.427, & k = 3 \\ 0, & \text{o.w.} \end{cases}$$

$$b_k = \begin{cases} -0.867, & k = 1 \\ 1.763, & k = 3 \\ 0, & \text{o.w.} \end{cases}$$

from (9.7)-(9.9)

$$\alpha_k = \begin{cases} \frac{1.8+0.867j}{2}, & k = 1 \\ \frac{1.8-0.867j}{2}, & k = -1 \\ \frac{-2.427-1.763j}{2}, & k = 3 \\ \frac{-2.427+1.763j}{2}, & k = -3 \\ 0, & \text{o.w.} \end{cases}$$

The plots are shown in Figure 257. The matlab code titled `hw10_p1.m` is in the course locker.



**Figure 257:** Problem 1

**9.31.** Plot the power spectrum of the functions in problem 9.11. Plot the graph in decibels.

**Solution.** Recall that the  $\alpha_n$  values are given by another problem.

(a)

$$\alpha_n = \begin{cases} \frac{.45}{2} & , n = \pm 1 \\ 0 & , \text{o.w.} \end{cases}$$

Therefore, the power spectrum in decibel is

$$P_s(n) = \begin{cases} 20 \log(.225) = -12.96 & , n = \pm 1 \\ -\infty & , \text{o.w.} \end{cases}$$

(b)